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A PAIR OF CALABI–YAU MANIFOLDS AS AN EXACTLY SOLUBLE SUPERCONFORMAL THEORY*

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We compute the prepotentials and the geometry of the moduli spaces for a Calabi–Yau manifold and its mirror. In this way we obtain all the sigma model corrections to the Yukawa couplings and moduli space metric for the original manifold. The moduli space is found to be subject to the action of a modular group which, among other operations, exchanges large and small values of the radius, though the action on the radius is not as simple as $R \to 1/R$. It is also shown that the quantum corrections to the coupling decompose into a sum over instanton contributions and moreover that this sum converges. In particular there are no "sub-instanton" corrections. This sum over instantons points to a deep connection between the modular group and the rational curves of the Calabi–Yau manifold. The burden of the present work is that a mirror pair of Calabi–Yau manifolds is an exactly soluble superconformal theory, at least as far as the massless sector is concerned. Mirror pairs are also more general than exactly soluble models that have hitherto been discussed since we solve the theory for all points of the moduli space.

1. Introduction

The discovery of mirror symmetry [1–3] among pairs of Calabi–Yau manifolds goes a long way towards resolving a long standing puzzle. A Calabi–Yau manifold $\mathcal{M}$ possesses a certain number of parameters. These are parameters associated with the structure of $\mathcal{M}$ as a complex manifold and parameters corresponding to the deformations of the Kähler metric of $\mathcal{M}$. These parameters, which are related to the cohomology of $\mathcal{M}$, give rise to families and antifamilies of particles in the effective low-energy theory that results from compactification of the string. The parameters corresponding to deformations of the complex structure are related to the cohomology group H^{21} of (2, 1)-forms while the parameters corre-

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sponding to deformations of the Kähler form correspond to the group H^{11} of $(1,1)$-forms. The Yukawa couplings of the low-energy theory correspond to certain cubic forms on the cohomology ring [4]. There are no couplings between the two different sorts of parameters, so the Yukawa couplings come in two types. The puzzle has been that the two types of couplings are very different both at a mathematical level and with regard to renormalization. The couplings corresponding to the complex structure parameters vary with the parameters and are not renormalized either in loops or by instantons. By contrast the couplings corresponding to the Kähler class are topological numbers that are integers in an appropriate basis and which *are* renormalized by instantons [5–7]. In this sense one might describe analysis based on a Calabi–Yau manifold as being "half exact". However, since both $\mathcal{M}$ and its mirror $\mathcal{W}$, for which the roles of the two types of parameters are exchanged, correspond to the same superconformal theory, one can combine the calculations and obtain exact results. One can compute both types of Yukawa couplings by calculating the couplings for the complex structure parameters of $\mathcal{M}$ and then computing the remaining couplings, complete with their sigma model corrections, by computing the couplings corresponding to the complex structure parameters of $\mathcal{W}$. The suggestion that Calabi–Yau manifolds should arise in mirror pairs was made by Dixon and Gepner [8] and by Lerche et al. [9]. The latter paper also uncovered the chiral ring structure of the superconformal theories. It is the identification of the chiral ring of the superconformal theory with the cohomology ring of $(2,1)$-forms [10] on the Calabi–Yau manifold that enables us to perform exact calculations by means of geometrical methods.

Of course if the Yukawa couplings were all we could compute then the results would not be very significant since we need also to be able to compute the metric on the parameter space, which appears in the kinetic terms of the sigma model, in order to correctly normalize the fields. Fortunately an extension of the nonrenormalization theorem that ensures that the superpotential does not receive sigma model corrections enables us to compute also the kinetic terms. The observation that the existence of mirror manifolds permits the calculation of both types of couplings has been made independently by Greene and Plesser [2], and also in the interesting article by Aspinwall et al. [3], who consider a mirror pair of manifolds with $\chi = \pm 40$ that has several parameters and show that, in an appropriately defined large complex structure limit, the Yukawa couplings for the complex structure parameters of the mirror manifold coincide with the topological couplings of the original manifold. One of the new elements of the present work is that we are able to solve, in the context of a particular example, for the Yukawa couplings and the metric on the parameter space for all values of the complex structure.

This article is devoted to a discussion of these issues in the context of the solution of the conformal field theory of a particular example of a mirror pair of Calabi–Yau manifolds. We take for the manifold $\mathcal{M}$ the quintic threefold $\mathbb{P}_4(5)$, which has $b_{11} = 1$, $b_{21} = 101$ and Euler number -200. The mirror $\mathscr{W}$ of this manifold is known in virtue of a construction due to Greene and Plesser [2]. The mirror $\mathscr{W}$ has $b_{11} = 101$, $b_{21} = 1$ and Euler number $+200$. What we do here is calculate the prepotential for the complex structure parameter of $\mathscr{W}$. By mirror symmetry this yields the fully corrected prepotential for the original manifold $\mathscr{M}$. Although we concentrate on a specific and simple case we believe that many features of our results are of general validity.

Since the present work is somewhat beset by detail we list here the salient results:

(i) The metric and the Yukawa couplings are computed complete with all sigma model corrections for all points in the parameter space. There is a particular value of the parameter for which the appropriate conformal field theory is the Gepner model 3^5 and for this value we find agreement with the known coupling for this model.

(ii) It is found that a modular group, Γ, acts on the parameter space. Among other operations Γ exchanges large and small values of the radius. This is of interest because it is relevant to the conjectured existence of a minimum fundamental length in string theory. The existence of a modular group has been noted previously for the case of orbifolds [11], and some consequences of modular invariance and the possibility that Calabi–Yau manifolds would also be subject to a modular group has been examined in a number of papers [12]. The modular group Γ however is not the group $SL(2, \mathbb{Z})$, as had previously been anticipated in the literature, and the operation is not as simple as $R \to 1/R$.

(iii) The exact Yukawa coupling admits a decomposition into a sum over instantons. The sum converges to the exact value so there are no "sub-instanton" contributions to the coupling or prepotential. Moreover the decomposition of the coupling into a sum over instanton contributions seems to indicate a deep connection between the automorphic functions of the modular group and the rational curves (instantons) of $\mathscr{M}$. As an illustration of this we seem to be able to read off from our results the numbers of rational curves of each degree. This is quite likely of mathematical interest.

(iv) We abstract from the particular pair of Calabi–Yau manifolds studied here an expression for the fully corrected Yukawa coupling, which we conjecture to be of general validity. The fundamental object from which the corrected coupling derives is not so much the "bare" manifold but rather a "quantum manifold" consisting of the bare manifold together with its rational curves. In physics-speak this is just the statement that the quantum manifold is the bare manifold together with all world-sheet instantons. However this statement is rendered more precise and seems to be in line with recent developments in mathematics (for a review see [13]).

The layout of this paper is as follows: in sect. 2 we review the construction of the mirror $\mathscr{W}$ and discuss the rudiments of the geometry of its space of complex structures. A feature that is important for the three-dimensional case is the existence of finite points in the parameter space corresponding to singular

Calabi–Yau manifolds. We describe here also the large complex structure limit of $\mathscr{M}$. The detailed structure of the metric and Yukawa couplings derive from a discussion of the homology of $\mathscr{W}$ and a computation of the holomorphic three-form from which the prepotential is constructed. This is done in sect. 3 and the metric and couplings are derived from the prepotential in sect. 4. We turn in sect. 5 to a comparison of the metric and couplings with the "bare" quantities appropriate to $\mathscr{M}$. For the couplings we identify the quantum corrections with the contributions of instantons. For the metric, in addition to the exponentially small terms, there is also a loop correction similar to the "four-loop term" found in another context by Grisaru et al. [14]. We also extend the standard nonrenormalization theorem to show that the four-loop term is the only loop term to affect the prepotential. In sect. 6 we present a speculative proposal for a mechanism to achieve a small breaking of supersymmetry at low energies. We have included this proposal here even though it is logically separate from the issue of mirror symmetry because the mechanism is based on a non-Kähler resolution of a conifold, and the conifold that arises in the study of the mirror manifold of $\mathbb{P}_4(5)$ is of precisely the type to which such a non-Kähler resolution is appropriate. Finally two appendices deal with a more detailed description of the homology of $\mathscr{W}$ and with further properties of the periods of the holomorphic three-form.

2. The mirror of $\mathbb{P}_4(5)$

Let $\mathsf{M} = \mathbb{P}_4(5)$ be the family of manifolds that can be represented as quintic hypersurfaces in $\mathbb{P}_4$. There are 101 parameters associated with the complex structure of these manifolds, which in this case can all be thought of as the coefficients of the quintic polynomial (see e.g. ref. [10], and references to the original literature cited therein). There is also one parameter associated with the choice of Kähler class which in this case can be thought of as the radius of the $\mathbb{P}_4$. To construct the mirror manifold [2] we start with a one-parameter subfamily M_1 of quintic hypersurfaces given by the polynomials

$$p = \sum_{k=1}^{5} x_k^5 - 5\psi \prod_{k=1}^{5} x_k . \tag{2.1}$$

These hypersurfaces are invariant under the symmetry group* generated by

$$\begin{aligned} g_0 &= (1,0,0,0,4)\,, \\ g_1 &= (0,1,0,0,4)\,, \\ g_2 &= (0,0,1,0,4)\,, \\ g_3 &= (0,0,0,1,4)\,, \end{aligned} \tag{2.2}$$

* We owe this choice of generators to B.R. Greene.

where the ith entry is the power of the fifth root of unity multiplying the ith coordinate. For example, g_1 represents the $\mathbb{Z}_5$ action

$$(x_1, x_2, x_3, x_4, x_5) \to (x_1, \alpha x_2, x_3, x_4, \alpha^4 x_5)$$

where $\alpha = e^{2\pi i/5}$. In virtue of the fact that the product of all the g's multiplies the homogeneous coordinates by a common phase, only three of these are independent so we can choose to work with g_1, g_2 and g_3, say. Note also that the quintic polynomial (2.1) is in fact the most general quintic invariant under these identifications. A family W of mirror manifolds is then obtained by taking the quotient of each Calabi–Yau manifold $\mathscr{M}_\psi$ in M_1 by the group $\mathbb{Z}_5^3$ generated by g_1, g_2 and g_3.

The procedure for dividing out by the $\mathbb{Z}_5$'s involves cutting out the curves and points of the manifold which are left invariant by the symmetries. After making the $\mathbb{Z}_5^3$ identifications, the curves and points are replaced by their smooth equivalents. The action of the $\mathbb{Z}_5^3$ has the 10 fixed curves

$$C_{ijk}: \quad x_i^5 + x_j^5 + x_k^5 = 0, \qquad i, j, k \text{ distinct}.$$

Each of these curves is a $\mathbb{P}_2(5)$ and is invariant under a $\mathbb{Z}_5$ subgroup. These fixed curves meet in the 10 fixed points

$$p_{ij}: \quad x_i^5 + x_j^5 = 0, \qquad i, j \text{ distinct}$$

(there are in fact only ten of these points owing to the identifications (2.2)), each being left invariant by a $\mathbb{Z}_5 \times \mathbb{Z}_5$ subgroup. Each fixed curve contains 3 fixed points, and 3 fixed curves meet in each of the fixed points. We will take care of the curves and the points separately, so we need to know the Euler number of the curves less the points. This is simply

$$\chi(\mathbb{P}_2(5)) - 15 = -25.$$

Also, we need to know how the $\mathbb{Z}_5$'s act on the curves. One leaves the curve invariant and the other two identify it with itself. Thus we calculate the Euler number of the mirror manifold to be

$$\chi = \frac{-200 - 10 \times 5 - 10 \times (-25)}{5 \times 5 \times 5} + 10 \times \frac{(-25) \times 5}{25} + 10 \times \frac{5 \times 25}{5} = +200.$$

A curious fact is that the Euler number of $\mathbb{P}_4(5)$ minus the fifty points and ten curves is zero*.

* P.S. Aspinwall informs us that this is a general feature of constructing a mirror manifold by starting with a manifold $\mathscr{M}$ and dividing by a symmetry group. The noncompact manifcld that remains after removing the fixed points and fixed curves has Euler number zero.

One naturally wonders if there are other ways of presenting this manifold. There are in fact two ways of constructing the manifold as a hypersurface in weighted $\mathbb{P}_4$'s. If we set

$$(x_1, x_2, x_3, x_4, x_5) = \left(y_1 y_3^{1/5}, y_2^{4/5} y_5^{1/5}, y_3^{4/5} y_4^{1/5}, y_4^{4/5} y_2^{1/5}, y_5^{4/5}\right),$$

the transformation being well defined in virtue of the identifications (2.2) (indeed the transformation was chosen so as to incorporate the identifications in a natural way), then the defining equation becomes

$$p = y_1^5 y_3 + y_2^4 y_5 + y_3^4 y_4 + y_4^4 y_2 + y_5^4 - 5\psi \prod_{k=1}^{5} y_k ,$$

and on reflection we see that the manifold is $\mathbb{P}_4^{(41,48,51,52,64)}(256)$. Similarly, one can also show* that the mirror manifold is also $\mathbb{P}_4^{(51,60,64,65,80)}(320)$ (thereby showing that the two weighted hypersurfaces are biholomorphic). In this case the coordinate transformation takes the form

$$(x_1, x_2, x_3, x_4, x_5) = \left(w_1 w_4^{1/5}, w_2^{4/5} w_5^{1/5}, w_3, w_4^{4/5} w_2^{1/5}, w_5^{4/5}\right),$$

and the polynomial is

$$p = w_1^5 w_4 + w_2^4 w_5 + w_3^5 + w_4^4 w_2 + w_5^4 - 5\psi \prod_{k=1}^{5} w_k .$$

These weighted projective spaces appear in the tables of ref. [16] and have Euler number $+200$ and $b_{11} = 101$. We find it most convenient to work with the original form (2.1) of the polynomial p, though we have to bear in mind that this form refers to a covering space of $\mathscr{W}$. To get to $\mathscr{W}$ itself we must make the identifications (2.2).

One of our aims is to describe the space of ψ's, that is the space W of complex structures of $\mathscr{W}_\psi$. The first thing to note is that ψ and $\alpha\psi$ correspond to the same complex structure since the replacement $\psi \to \alpha\psi$ is equivalent to the coordinate transformation

$$(x_1, x_2, x_3, x_4, x_5) \to \left(\alpha^{-1} x_1, x_2, x_3, x_4, x_5\right); \tag{2.3}$$

in other words, we learn that the true coordinate is ψ^5.

To describe the geometry of W, it is important to note that there are special values of ψ for which $\mathscr{W}_\psi$ is singular. This occurs when the quintic (2.1) fails to be

* For a more general discussion of these techniques see ref. [15].

transverse. That is the case when the five equations

$$\frac{\partial p}{\partial x_k} = 0, \qquad k = 1,\dots,5 \tag{2.4}$$

are simultaneously satisfied. These equations imply that

$$x_1^5 = x_2^5 = \dots = x_5^5 = \psi \prod_{k=1}^{5} x_k, \tag{2.5}$$

whence

$$\prod_{k=1}^{5} x_k^5 = \psi^5 \prod_{k=1}^{5} x_k^5.$$

If ψ is finite then none of the x_i may be zero for if one were zero then by eq. (2.5) all would be zero, which is not allowed. It follows that (2.4) can only be satisfied if $\psi^5 = 1$. If we take $\psi = 1$ say, then returning to (2.5), we see that

$$x_k = \alpha^{n_k}, \qquad \sum n_k = 0.$$

These points are all identified under the identifications (2.2) so that $\mathcal{W}$ at $\psi = 1$ has only one singular point which we may as well take to be the point $(1,1,1,1,1)$. The singularity is a node, that is a point for which p and $\mathrm{d}p$ both vanish but the matrix of second derivatives is nonsingular. For these values of ψ, the corresponding $\mathcal{W}$ is a conifold [17]. This type of singular Calabi–Yau manifold has been described in some detail in refs. [17, 18]. For our present purpose it suffices to recall that a neighborhood of the node is locally a cone with base $S^2 \times S^3$. For values of ψ near $\psi = 1$, say, the situation is as in fig. 1. There is an S^3 which shrinks to zero as $\psi \to 1$.

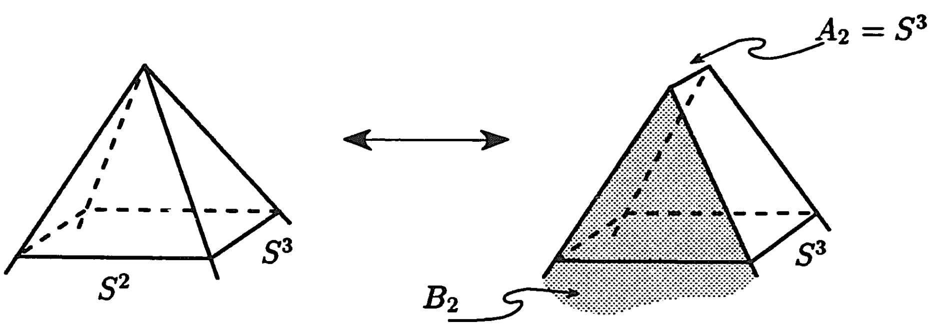

Fig. 1. The singular point of the conifold has a neighborhood that is a cone with base $S^2 \times S^3$. For $\psi - 1$ small but nonzero the node is replaced by a sphere of radius $O((\psi - 1)^{1/2})$.

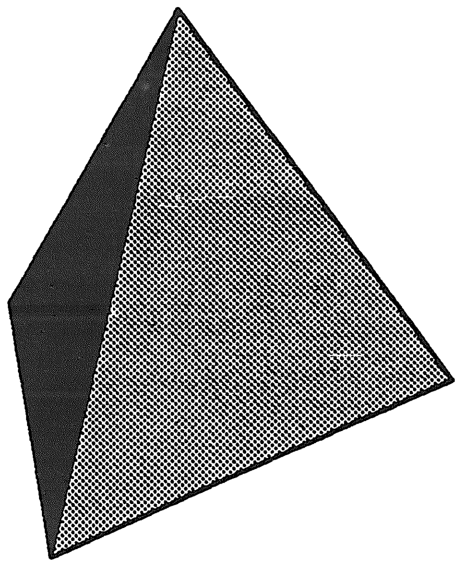

Fig. 2. A heuristic sketch, drawn as a tetrahedron, of the large complex structure limit of $\mathcal{W}$. Before identification under the $\mathbb{Z}_5^3$, the space consists of 5 $\mathbb{P}_3$'s meeting in 10 $\mathbb{P}_2$'s meeting in 10 $\mathbb{P}_1$'s meeting in 5 points.

The value $\psi = \infty$ corresponds to the singular quintic

$$\mathcal{W}_\infty: \quad x_1 x_2 x_3 x_4 x_5 = 0 ,$$

which, before identification under the $\mathbb{Z}_5^3$, consists of 5 $\mathbb{P}_3$'s meeting in 10 $\mathbb{P}_2$'s meeting in 10 $\mathbb{P}_1$'s meeting in 5 points. A lower-dimensional heuristic sketch is given in fig. 2. We shall see later that $\mathcal{W}_\infty$ is the large complex structure limit of $\mathcal{W}_\psi$ and is the mirror of the large-radius limit of $\mathcal{M}$.

2.1. RUDIMENTS OF THE HOMOLOGY

The structure of the moduli space of a Calabi–Yau manifold reflects the homology of the manifold. Recall that the complex structure of the manifold can be described by giving the periods of the holomorphic three-form over a canonical homology basis [17, 19, 20]. More precisely, we can proceed, for the case under consideration for which $b_{21}(\mathcal{W}) = 1$ and $b_3(\mathcal{W}) = 4$, as follows. We choose a symplectic basis $(\mathrm{A}^1, \mathrm{A}^2, \mathrm{B}_1, \mathrm{B}_2)$ for $\mathrm{H}_3(\mathcal{W}, \mathbb{Z})$ such that

$$\mathrm{A}^a \cap \mathrm{B}_b = \delta^a{}_b , \qquad \mathrm{A}^a \cap \mathrm{A}^b = 0 , \qquad \mathrm{B}_a \cap \mathrm{B}_b = 0 . \tag{2.6}$$

Let (α_a, β^b) be the cohomology basis dual to that above so that

$$\int_{\mathrm{A}^a} \alpha_b = \delta^a{}_b , \qquad \int_{\mathrm{B}_a} \beta^b = \delta_a{}^b ,$$

with other integrals vanishing. Then it follows that

$$\int_{\mathscr{W}} \alpha_a \wedge \beta^b = \delta_a{}^b, \qquad \int_{\mathscr{W}} \alpha_a \wedge \alpha_b = 0, \qquad \int_{\mathscr{W}} \beta^a \wedge \beta^b = 0.$$

Important in the following is the holomorphic three-form Ω. Being a three-form, Ω may be expanded in terms of the basis

$$\Omega = z^a \alpha_a - \mathscr{G}_a \beta^a.$$

The coefficients $(z^a, \mathscr{G}_b)$ are the periods of Ω, so called because they are given by the integrals of Ω over the homology basis,

$$z^a = \int_{\mathrm{A}^a} \Omega, \qquad \mathscr{G}_b = \int_{\mathrm{B}_b} \Omega. \tag{2.7}$$

We have more periods than parameters. For the present case we have $b_3 = 4$ periods but we know there is only 1 $(= b_{21})$ parameter for the complex structure. We therefore choose to regard the $\mathscr{G}_b$ as being functions of the z^a, $\mathscr{G}_b = \mathscr{G}_b(z^a)$. This leaves us with the z^a as independent periods. We still have one parameter too many but this turns out to fit in quite well. The scale of Ω is not defined. In reality Ω is a section of a line bundle over the moduli space [21]. There is a gauge invariance associated with Ω due to the fact that Ω is undefined up to multiplication by an arbitrary holomorphic function of the parameter

$$\Omega(\psi) \to f(\psi)\Omega(\psi), \tag{2.8}$$

and we can regard the two z^a as projective coordinates for Ω. The $\mathscr{G}_a$ and Ω are then homogeneous of degree one as functions of the z^a,

$$\Omega(\lambda z) = \lambda\Omega(z), \qquad \mathscr{G}_b(\lambda z) = \lambda\mathscr{G}_b(z),$$

and it can be shown [17, 20] that the $\mathscr{G}_b$ are the gradients of a prepotential $\mathscr{G}(z)$ that is homogeneous of degree two,

$$\mathscr{G}_a = \frac{\partial\mathscr{G}}{\partial z^a}, \qquad \mathscr{G}(\lambda z) = \lambda^2\mathscr{G}(z).$$

For the situation at hand for which the manifold can be represented by a single polynomial, there is a well-known representation for the holomorphic three-form [4, 10]. We make the specific gauge choice

$$\Omega = 5\psi \frac{x_5\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial p/\partial x_4}. \tag{2.9}$$

We shall have more to say about the gauge invariance in the following. For the present however, (2.9) is a convenient choice because the replacement $\psi \to \alpha\psi$ can be undone by the coordinate transformation (2.3). Thus

$$\Omega(\alpha\psi) = \Omega(\psi).$$

As a choice of basis we take A^2 to be the S^3 that shrinks to zero as $\psi \to 1$. The cycle B_2 is then a three-cycle that intersects this S^3 in a point. The "tip" of B_2, the rest of which lies outside the neighborhood, is indicated by the shading in fig. 1. The other basis cycles A^1 and B_1 do not intersect the S^3 and can be taken to lie outside the neighborhood. We are concerned about the monodromy of the basis about $\psi = 1$, say. Under transport about $\psi = 1$, the S^3, being unambiguously defined for each value of ψ, will return to the same cycle. The cycle B_2 on the other hand is really only defined as being the dual of the A^2, so nothing prevents B_2 from aquiring a multiple of A^2. Thus

$$A^2 \to A^2, \qquad B_2 \to B_2 + nA^2,$$

for some integer n. The remaining cycles A^1 and B_1 are remote from the node and so are unaffected by this process. The monodromy of the basis induces a monodromy for the periods (2.7),

$$\begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ z^1 \\ z^2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ z^1 \\ z^2 \end{pmatrix}. \tag{2.10}$$

We shall see later that the periods, as functions of ψ, satisfy a linear differential equation which has a singularity at $\psi = 1$. The relation (2.10) can be understood to describe the monodromy of the solutions of the differential equation about the singular point. Such monodromy is familiar, expressing the fact that some of the solutions of a differential equation in the neighborhood of a singularity contain logarithms and hence are multivalued.

2.2. THE MODULAR GROUP

It is convenient to adopt a matrix notation and to define a period vector

$$\Pi = \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \\ z^1 \\ z^2 \end{pmatrix}. \tag{2.11}$$

We have just argued that under transport around $\psi = 1$ the period vector undergoes a transformation $\Pi \to T\Pi$. The matrix T is symplectic, since the transformation must preserve the intersection numbers (2.6), and it is also integral, since the homology basis is integral. We have observed also that ψ and $\alpha\psi$ correspond to the same manifold. It follows that

$$\Pi(\alpha\psi) = A\Pi(\psi),$$

with A an integral symplectic matrix such that $A^5 = 1$. We will obtain the precise form of A in sect. 3 but for the present it suffices to remark that it is not the identity, neither is T, since we will also find that the integer in (2.10) has the value unity. The two matrices A and T generate a modular group Γ that acts on the period vectors, and also on the universal covering space of W, which may be identified with the upper half-plane in such a way that Γ acts by hyperbolic isometries. We shall denote the operations of replacing ψ by $\alpha\psi$ and of analytic continuation about $\psi = 1$ by $\mathscr{A}$ and $\mathscr{T}$. These operations are represented by the matrices A and T. However, care is required when composing the operations since the matrices compose "backwards". For example

$$\mathscr{T}(\mathscr{A}\Pi) = \mathscr{T}(A\Pi) = A(\mathscr{T}\Pi) = AT\Pi.$$

Transport about $\psi = \alpha^k$ corresponds to the operation $\mathscr{A}^k\mathscr{T}\mathscr{A}^{-k}$ and hence to the matrix

$$T_k = A^{-k}TA^k, \tag{2.12}$$

and the matrix T_∞ corresponding to transport about $\psi = \infty$ follows from the observation that a sum of loops around the fifth roots of unity and around infinity is contractible. Hence

$$T_\infty^{-1} = T_4T_3T_2T_1T = (AT)^5. \tag{2.13}$$

Thus there are no new generators corresponding to these operations. ψ^5 is a modular invariant of Γ but we can pass to a modular parameter $\gamma = \gamma(\psi)$ such that the action of Γ on γ is represented by 2×2 matrices acting on the upper half γ-plane. Since $\mathscr{A}^5 = 1$ we represent it as

$$\mathsf{A} = -\begin{pmatrix} \cos(2\pi j/5) & \sin(2\pi j/5) \\ -\sin(2\pi j/5) & \cos(2\pi j/5) \end{pmatrix},$$

where the minus sign is of no consequence in virtue of the projectivity of the representation, and j is an integer, $1 \leqslant j \leqslant 4$. Since $\mathscr{T}^k$ is never the identity for any $k \neq 0$, $\mathscr{T}$ acts on the upper half-plane as a "deck transformation" of infinite order. Some composition of $\mathscr{T}$ with powers of $\mathscr{A}$ may therefore be represented by

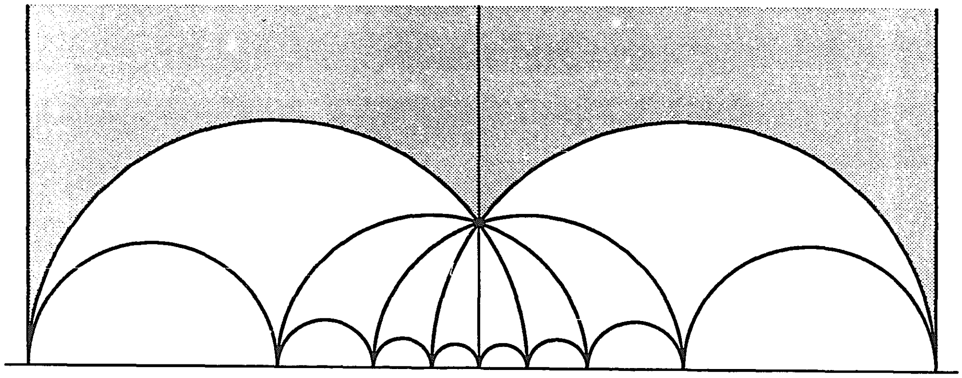

Fig. 3. The fundamental regions for the group generated by $\mathscr{A}$ and $\mathscr{T}$ consist of a pair of triangles. The circles intersect the real axis where $\gamma = \tan(r\pi/10)$, $r = -4,\ldots,4$. If we take the shaded region to be the fundamental region, the points $\psi = (0, 1, \infty, \alpha)$ map to $\gamma = (i, \tan(2\pi/5), \infty, -\tan(2\pi/5))$, respectively.

a matrix of the form

$$\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}.$$

The magnitude of the translation is fixed by the fact that a fundamental region for the normal subgroup of Γ generated by $\mathscr{T}$ is a ten-sided polygon whose sides are permuted by $\mathscr{A}$. This amounts to considering the images under $\mathscr{A}$, $\mathscr{A}^2$, $\mathscr{A}^3$ and $\mathscr{A}^4$ of the imaginary axis and then demanding compatibility with $\mathscr{T}$. Fundamental domains of Γ are illustrated in fig. 3.

The explicit form of $\gamma(\psi)$ may be found by seeking a map that maps the entire ψ^5-plane into a pair of triangles. This is a standard procedure in the theory of automorphic functions involving triangle functions [22]. If the fundamental region for γ is taken to be the shaded region in fig. 3 then the relation is

$$\begin{aligned}\gamma &= i\left(\frac{Z_1 - \alpha^2 Z_2}{Z_1 + \alpha^2 Z_2}\right)\\ &= \frac{i}{\pi}\tan(2\pi/5)\left\{\log(\psi^5) - i\pi + \frac{\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})}{(n!)^2\psi^{5n}}\left[2\Psi(n+1) - \Psi(n+\tfrac{2}{5}) - \Psi(n+\tfrac{3}{5})\right]}{\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})}{(n!)^2\psi^{5n}}}\right\},\end{aligned} \tag{2.14}$$

where the second equality is valid for $|\psi| > 1$, Ψ denotes the digamma function, and Z_1 and Z_2 are defined in terms of Gauss' series,

$$Z_1=\frac{\Gamma^2(\frac{2}{5})}{\Gamma(\frac{4}{5})}\,{}_2F_1(\tfrac{2}{5},\tfrac{2}{5};\tfrac{4}{5};\psi^5),\qquad Z_2=\psi\frac{\Gamma^2(\frac{3}{5})}{\Gamma(\frac{6}{5})}\,{}_2F_1(\tfrac{3}{5},\tfrac{3}{5};\tfrac{6}{5};\psi^5).$$

We may now complete the specification of the matrices A and T. Recall that $\mathscr{A}$ maps ψ to $\alpha\psi$ and $\mathscr{T}$ transports ψ about $\psi=1$. It is straightforward to compute the effect of these operations on γ in virtue of (2.14) and the standard analytic continuation formulae for the hypergeometric function*. If we require that the action of A is to rotate fundamental regions by $2\pi/5$ about the fixed point $\gamma=i$, then it turns out that $j=3$ and we find

$$\mathsf{A}=\begin{pmatrix}\cos\frac{\pi}{5} & \sin\frac{\pi}{5}\\ -\sin\frac{\pi}{5} & \cos\frac{\pi}{5}\end{pmatrix},\qquad \mathsf{AT}=\begin{pmatrix}1 & -2\tan(2\pi/5)\\ 0 & 1\end{pmatrix}.$$

These relations define the representation. It is easy to check also that

$$\mathsf{A}^3\mathsf{T}\mathsf{A}^{-3}=\begin{pmatrix}1 & 0\\ -2\tan(2\pi/5) & 1\end{pmatrix},$$

from which we see that the various monodromy matrices are conjugates of the matrix on the right.

The upper half-plane may be mapped to the interior of the unit circle by the transformation

$$\xi=\alpha^3\left(\frac{1+i\gamma}{1-i\gamma}\right)=\frac{Z_2}{Z_1},$$

the fundamental regions of Γ are then as illustrated in fig. 4. In this representation the action of $\mathscr{A}$ is simply that of multiplication by α.

We defer further discussion of Γ to sect. 5, save to observe that Γ which might be termed the quantum modular group is not $SL(2,\mathbb{Z})$, which had previously been suggested as the modular group, based on its action on the "bare" moduli space of $\mathbb{P}_4(5)$.

In order to find the explicit form of the metric we need to evaluate explicitly the periods of Ω. It is to this that we now turn.

* If we set

$$f(a;\zeta)=\frac{\Gamma^2(a)}{\Gamma(2a)}\,{}_2F_1(a,a;2a;\zeta),$$

then the essential relation is

$$\mathscr{T}f(a;\zeta)=f(a;\zeta)-i\tan\pi a\{f(a;\zeta)-\zeta^{1-2a}f(1-a;\zeta)\}.$$

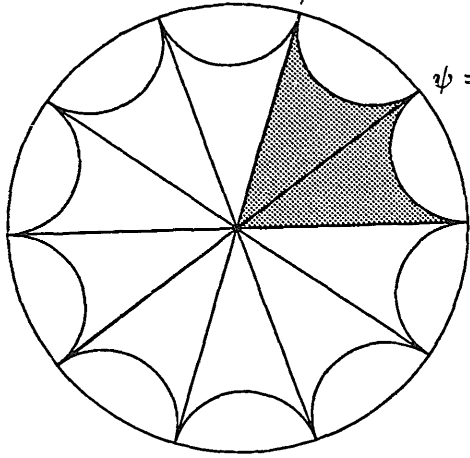

Fig. 4. Fundamental regions of Γ in the unit disk.

3. The periods

We begin by specifying more carefully the A^2 and B_2 of our symplectic basis. A^2 is the S^3 that degenerates to zero at the conifold point $\psi = 1$ and B_2 is a certain torus associated with the degeneration of the manifold as $\psi \to \infty$.

$$A^2 = \{x_k | x_5 = 1, \quad x_i \text{ real}, \quad i = 1,2,3, \\ x_4 \text{ given by the branch of } p(x) = 0 \text{ that is an } S^3 \text{ as } \psi \to 1\}. \tag{3.1}$$

Consider first the case that $\psi - 1$ is real, positive and small. Then by writing

$$x_1 = 1 + \frac{y_1}{\sqrt{10}} + \frac{y_2}{5} + \frac{y_4}{\sqrt{50}},$$

$$x_2 = 1 + \frac{y_1}{\sqrt{10}} - \frac{y_2}{5} + \frac{y_4}{\sqrt{50}},$$

$$x_3 = 1 + \frac{y_1}{\sqrt{10}} + \frac{y_3}{5} - \frac{y_4}{\sqrt{50}},$$

$$x_4 = 1 + \frac{y_1}{\sqrt{10}} - \frac{y_3}{5} - \frac{y_4}{\sqrt{50}},$$

and keeping lowest-order terms, we see that

$$\sum_{k=1}^{4} y_k^2 = 5(\psi - 1), \tag{3.2}$$

which is manifestly an S^3. The qualification in the definition of A^2 concerning the branch is necessary, owing to the fact that the surface $p(x)=0$ for x_k real has disconnected components, as is easily seen by observing that the line

$$(x_1, -x_1, 0, -1, 1)$$

is remote from the point $(1,1,1,1,1)$ and yet identically satisfies $p(x)=0$.

On integrating the three-form Ω over A^2 we find

$$\begin{aligned} z^2(\psi) &= \int_{A^2} \Omega \\ &= \frac{4\pi^2}{5^{3/2}}(\psi - 1) + \dots . \end{aligned} \tag{3.3}$$

The leading term may be obtained by integrating the zeroth-order locus (3.2), and the higher terms may be calculated systematically by means of an iteration scheme. Shortly we shall give an exact expression for $z^2(\psi)$ in terms of hypergeometric functions.

For B_2 we take the cycle

$$\begin{aligned} B_2 = \{ x_k | x_5 = 1, \ |x_1| = |x_2| = |x_3| = \delta, \\ x_4 \text{ given by the solution to } p(x) = 0 \text{ that tends to zero as } \psi \to \infty \}. \end{aligned} \tag{3.4}$$

To understand the condition on the branch of the solution, set $x_4 = (\psi x_1 x_2 x_3)^{1/4} \xi$ and write the equation $p=0$ in the form

$$\xi = \frac{(1 + x_1^5 + x_2^5 + x_3^5)}{5(\psi x_1 x_2 x_3)^{5/4}} + \frac{\xi^5}{5}, \tag{3.5}$$

from which it is clear that to leading order as $\psi \to \infty$, there is a solution for ξ given by the first term on the right-hand side of eq. (3.5). Thus there is a branch for x_4 with $x_4 = O(\psi^{-1})$ as $\psi \to \infty$ for fixed x_1, x_2, x_3. On the other hand, by rearranging the equation $p=0$ into the form

$$\xi^4 = 5 - \frac{(1 + x_1^5 + x_2^5 + x_3^5)}{(\psi x_1 x_2 x_3)^{5/4} \xi},$$

we see that there are four branches for x_4 that are $O(\psi^{1/4})$ as $\psi \to \infty$ for fixed (x_1, x_2, x_3). At this stage it is far from clear that, as defined, B_2 meets A^2 in a single point. Showing that it does, requires a more detailed discussion of the homology than we wish to give here, so we defer the demonstration of this fact to appendix A. We turn instead to a computation of the periods.

We have

$$\mathscr{G}_2 \stackrel{\text{def}}{=} \int_{\mathrm{B}_2} \Omega$$

$$= \int_{\mathrm{B}_2} \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}{x_1 x_2 x_3 - \psi^{-1} x_4^4}\,. \tag{3.6}$$

A comment regarding the orientations of the cycles A^2 and B_2 is necessary here. The cycles do not possess an intrinsically defined orientation so a choice must be made. We have already implicitly fixed the orientation of A^2 by eq. (3.3). This in turn fixes the orientation of B_2 since we require that $\mathrm{A}^2 \cap \mathrm{B}_2 = +1$. The choice made in the second of eqs. (3.6) is consistent with this. As $\psi \to \infty$ we see that the term involving ψ can be neglected and hence

$$\mathscr{G}_2 \to (2\pi i/5)^3,$$

the factor of 5^{-3} arising from the $\mathbb{Z}_5^3$ identifications. For ψ large the integrand can be expanded in powers of (ξ^4/ψ). ξ can itself be computed as a power series in ψ^{-1} in virtue of an iteration based on eq. (3.5). The result is of the form

$$\mathscr{G}_2 = \sum_{n=0}^{\infty} a_n \int \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}{x_1 x_2 x_3} \frac{\left(1 + x_1^5 + x_2^5 + x_3^5\right)^{4n}}{\left(x_1^5 x_2^5 x_3^5 \psi^5\right)^n}\,.$$

We evaluate the integrals by residues. The only term in the quantity

$$\left(1 + x_1^5 + x_2^5 + x_3^5\right)^{4n}$$

that contributes is the term $x_1^{5n} x_2^{5n} x_3^{5n}$ which appears with coefficient $(4n)!/(n!)^4$. Thus

$$\mathscr{G}_2 = \left(\frac{2\pi i}{5}\right)^3 \sum_{n=0}^{\infty} a_n \frac{(4n)!}{(n!)^4 \psi^{5n}}\,.$$

The surprise is that when the a_n are calculated and substituted into this expression, we find

$$\mathscr{G}_2 = \left(\frac{2\pi i}{5}\right)^3 \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}\,, \tag{3.7}$$

and the series on the right converges for $|\psi| \geqslant 1$.

The function that appears on the right-hand side of eq. (3.7),

$$\varpi_0(\psi) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \qquad |\psi| > 1, \qquad 0 \leqslant \arg\psi < 2\pi/5, \qquad (3.8)$$

satisfies a linear differential equation of generalized hypergeometric type and it is useful to take advantage of this fact in order to write down a complete set of periods. The restriction on $\arg\psi$ anticipates that analytic continuation of $\varpi_0(\psi)$ will lead to branch cuts. The fact that periods of the type we are discussing satisfy linear differential equations with regular singular points is a very general property [23]. For the case at hand it is straightforward to check that

$$\left\{\frac{\mathrm{d}^4}{\mathrm{d}z^4} - \frac{2(4z-3)}{z(1-z)}\frac{\mathrm{d}^3}{\mathrm{d}z^3} - \frac{(72z-35)}{5z^2(1-z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{(24z-5)}{5z^3(1-z)}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{24}{625z^3(1-z)}\right\}\varpi_0 = 0, \qquad (3.9)$$

where $z = \psi^{-5}$. It is compelling to assume that all four periods satisfy this same equation, a fact that we shall assume for the present but verify shortly. Either directly from the equation or from the associated Riemann symbol

$$\mathscr{P}\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \frac{1}{5} & 0 & \\ 0 & \frac{2}{5} & 1 & \psi^{-5} \\ 0 & \frac{3}{5} & 2 & \\ 0 & \frac{4}{5} & 1 & \end{matrix}\right\}, \qquad (3.10)$$

we observe that the equation is of generalized hypergeometric type. Recall that the generalized hypergeometric equation of fourth order is

$$\{\vartheta(\vartheta + c_1 - 1)(\vartheta + c_2 - 1)(\vartheta + c_3 - 1) - z(\vartheta + a_1)(\vartheta + a_2)(\vartheta + a_3)(\vartheta + a_4)\}w = 0,$$

where $\vartheta = z\,\mathrm{d}/\mathrm{d}z$ and the associated Riemann symbol is

$$\mathscr{P}\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a_1 & 0 & \\ 1-c_1 & a_2 & 1 & z \\ 1-c_2 & a_3 & 2 & \\ 1-c_3 & a_4 & \Sigma c_j - \Sigma a_k & \end{matrix}\right\}.$$

The solutions of the differential equation (3.9) can be written in terms of the generalized hypergeometric function [22, 24]

$$
{}_4F_3(a_1,a_2,a_3,a_4;c_1,c_2,c_3;\zeta) = \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}
\times \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(a_3+n)\Gamma(a_4+n)}{\Gamma(c_1+n)\Gamma(c_2+n)\Gamma(c_3+n)} \frac{\zeta^n}{n!}
$$

In fact we have

$$
\varpi_0(\psi) = {}_4F_3\left(\tfrac{1}{5},\tfrac{2}{5},\tfrac{3}{5},\tfrac{4}{5};1,1,1;1/\psi^5\right), \tag{3.11}
$$

as is easily verified with the aid of the multiplication formula for the Γ-function in the form

$$
\Gamma(z)\Gamma(z+1/5)\Gamma(z+2/5)\Gamma(z+3/5)\Gamma(z+4/5) = (2\pi)^2 5^{1/2-5z}\Gamma(5z). \tag{3.12}
$$

A standard maneuver for finding the other solutions to the differential equation in terms of the hypergeometric function amounts here to changing variables from $1/\psi^5$ to ψ^5 and to extracting a factor of ψ^k, with $k=1$, 2, 3, or 4, from the Riemann symbol. Thus a set of four linearly independent solutions are

$$
\psi^k \mathscr{P}\left\{\begin{matrix} 0 & \infty & 1 & \\ 1/5-k/5 & k/5 & 0 & \\ 2/5-k/5 & k/5 & 1 & \psi^5 \\ 3/5-k/5 & k/5 & 2 & \\ 4/5-k/5 & k/5 & 1 & \end{matrix}\right\}
$$

$$
= \psi^k {}_4F_3\left(\frac{k}{5},\frac{k}{5},\frac{k}{5},\frac{k}{5};\overbrace{\frac{k+1}{5},\frac{k+2}{5},\frac{k+3}{5},\frac{k+4}{5}};\psi^5\right), \tag{3.13}
$$

where the overbrace signifies that the parameter that is unity is to be omitted. This basis is useful for some purposes, however we find it more convenient to base our development on the functions $\varpi_0(\alpha^k\psi)$. At first sight one might be tempted to conclude from eq. (3.11) that $\varpi_0(\psi)$ is a function of ψ^5 but this is not the case owing to the fact that there are branch cuts implicit in the definition of $\varpi_0(\psi)$. In

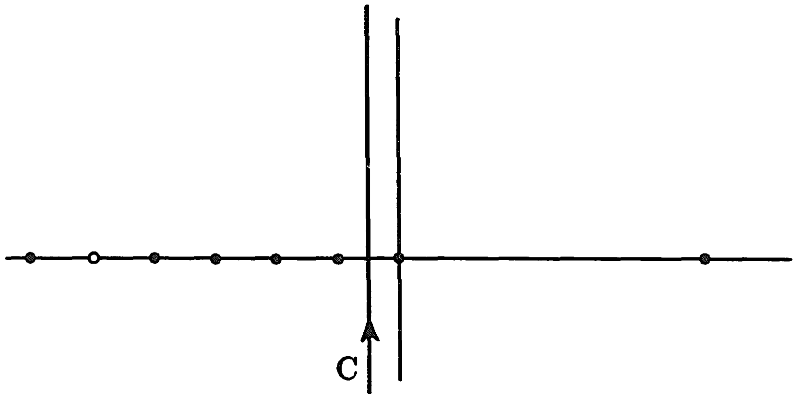

Fig. 5. The contour of integration for eq. (3.14) showing the poles of $\Gamma(-s)$ and of $\Gamma(5s+1)$.

order to clarify this and to find explicit expressions for $\varpi_0(\alpha^k\psi)$, we wish to analytically continue $\varpi_0(\psi)$ to a neighborhood of the origin. This is accomplished by introducing an integral representation of Barne's type. If $0<\arg\psi<2\pi/5$, we have

$$\varpi_0(\psi)=\frac{1}{2\pi i}\int_C \mathrm{d}s\,\frac{\Gamma(-s)\Gamma(5s+1)}{\Gamma^4(s+1)}\,\mathrm{e}^{i\pi s}(5\psi)^{-5s}, \tag{3.14}$$

with the contour as in fig. 5. The integrand has poles on the positive real axis for $s=0,1,2,\ldots$ due to the presence of the factor $\Gamma(-s)$. If in addition $|\psi|>1$, the contour can be closed to the right and (3.11) is recovered as a sum over residues. If on the other hand $|\psi|<1$, the contour can be closed to the left enclosing the poles of $\Gamma(5s+1)$. Thus we find

$$\varpi_0(\psi)=-\tfrac{1}{5}\sum_{m=1}^{\infty}\frac{\alpha^{2m}\Gamma(m/5)(5\psi)^m}{\Gamma(m)\Gamma^4(1-m/5)},\qquad |\psi|<1. \tag{3.15}$$

We shall show presently that $\varpi_0(\psi)$ contains logarithms when $\psi^5=1$, so we must specify cuts associated with these terms. It is convenient to take $\varpi_0(\psi)$ to be analytic in the neighborhood of the origin, so we take the cuts to run radially outward from the points $\psi=\alpha^k$, $k=0,\ldots,4$. It is clear from the form of the differential operator that the functions

$$\varpi_j(\psi)\stackrel{\text{def}}{=}\varpi_0(\alpha^j\psi),\qquad j=0,\ldots,4 \tag{3.16}$$

all satisfy the differential equation. It is clear also from the series (3.15) that any four of these functions are linearly independent, the five $\varpi_j(\psi)$ being subject to the single relation

$$\sum_{j=0}^{4} \varpi_j(\psi) = 0. \tag{3.17}$$

We wish next to examine further the monodromy of this basis about the point $\psi = 1$. We have argued that the basis $(\mathscr{G}_a, z^b)$ transforms according to the rule (2.10) under transport about the point $\psi = 1$. It follows that the ϖ_j have the transformation rule

$$\left(\frac{2\pi i}{5}\right)^3 \varpi_j(\psi) \to \left(\frac{2\pi i}{5}\right)^3 \varpi_j(\psi) + c_j z^2(\psi), \tag{3.18}$$

where the c_j are a set of numerical coefficients. This transformation rule is equivalent to the assertion that the ϖ_j have the structure

$$\left(\frac{2\pi i}{5}\right)^3 \varpi_j(\psi) = \frac{c_j}{2\pi i} z^2(\psi)\log(\psi - 1) + f_j(\psi), \tag{3.19}$$

with $z^2(\psi)$ and $f_j(\psi)$ analytic for $|\psi - 1| < 1$. The period $z^2(\psi)$ corresponds to one of the indices which is unity in (3.10). The differential equation, as is easily verified, admits two solutions that are given by power series about $\psi = 1$. These are $z^2(\psi)$ and another series corresponding to the index 2. The other solutions contain logarithms owing to the fact that the index 1 is repeated. One might have anticipated a more complicated structure than (3.19), with the term multiplying the logarithm being a more general linear combination of the two solutions that have power series expansions. This however would not be consistent with (2.10). Checking that the transformation rule is indeed as in (3.19) amounts to checking the coefficient of $(\psi - 1)^2 \log(\psi - 1)$ in $\varpi_j(\psi)$. This will be subsumed in the following computation of the coefficients c_j.

Suppose $z^2(\psi)$ has the expansion

$$z^2(\psi) = \frac{4\pi^2}{5^{3/2}}\left\{(\psi - 1) + b(\psi - 1)^2 + \dots\right\} \tag{3.20}$$

about $\psi = 1$. Then from eq. (3.19) we have

$$\left(\frac{2\pi i}{5}\right)^3 \frac{\mathrm{d}^2}{\mathrm{d}\psi^2}\varpi_j(\psi) = -\frac{2\pi i}{5^{3/2}} c_j \left\{\frac{1}{\psi - 1} + 2b\log(\psi - 1) + \dots\right\}, \tag{3.21}$$

where the terms indicated by the ellipsis have finite limits as $\psi \to 1$. Thus we can calculate c_j and b by differentiating the series (3.15) and computing its leading behaviour as $\psi \to 1$. It is convenient to set

$$m = 5N + k, \qquad k = 0, \ldots, 4, \qquad N = 0, 1, 2, \ldots$$

for the variable of summation in eq. (3.15). Then in virtue of Stirling's formula

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-1/2} \mathrm{e}^{-z} \left\{ 1 + \frac{1}{12z} + \mathrm{O}(z^{-2}) \right\},$$

we find that as $\psi \to 1$

$$\frac{\mathrm{d}^2 \varpi_j}{\mathrm{d}\psi^2} \sim -\frac{5^{3/2}}{4\pi^2} \sum_{k=0}^{4} \alpha^{kj} (\alpha^k - 1)^4 \sum_N^{\infty} \left(\psi^{5N} + \frac{\psi^{5N}}{5N} \right).$$

By comparing this expression with eq. (3.21), we find $b = 1/2$ and

$$c_j = (1, 1, -4, 6, -4) \quad \text{for } j = (0, 1, 2, 3, 4).$$

The next step is to express $z^2(\psi)$ in terms of the basis $\varpi_j(\psi)$. The result is

$$z^2(\psi) = -\left(\frac{2\pi i}{5} \right)^3 (\varpi_1(\psi) - \varpi_0(\psi)). \tag{3.22}$$

To see this suppose x is real and $x > 1$. If ϵ is an infinitesimal, then from eq. (3.19)

$$\left(\frac{2\pi i}{5} \right)^3 (\varpi_1(x + i\epsilon) - \varpi_1(x - i\epsilon)) = -z^2(x).$$

However

$$\begin{aligned} \varpi_1(x - i\epsilon) &= \varpi_0(\alpha(x - i\epsilon)) \\ &= \varpi_0(x + i\epsilon), \end{aligned}$$

the last equality following from (3.8) which is valid for $0 \leqslant \arg \psi < 2\pi/5$. Thus eq. (3.22) holds for ψ's just above the cut that extends from 1 to ∞ (see fig. 6), and hence by analytic continuation for all ψ.

We now know z^2 and $\mathscr{G}_2$ in terms of the basis ϖ_j, and we wish to find similar expressions for z^1 and $\mathscr{G}_1$. It turns out, rather surprisingly, that we do not have to explicitly describe A^1 and B_1 if we proceed somewhat indirectly. We choose a specific basis of linearly independent ϖ_j and form a vector

$$\varpi \stackrel{\text{def}}{=} -\left(\frac{2\pi i}{5} \right)^3 \begin{pmatrix} \varpi_2 \\ \varpi_1 \\ \varpi_0 \\ \varpi_4 \end{pmatrix}.$$

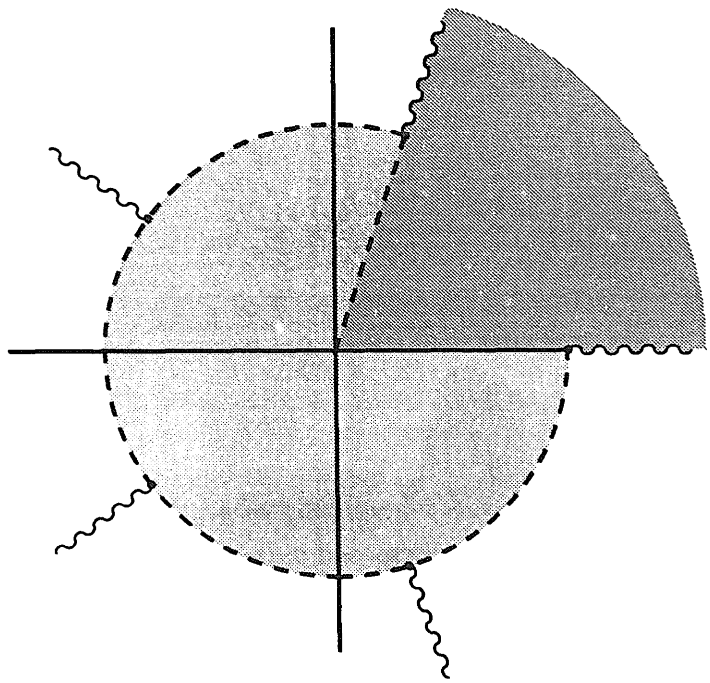

Fig. 6. The cut ψ-plane. The fundamental region is indicated by the heavier shading. The series for the $\varpi_j(\psi)$ given by eqs. (3.16) and (3.15) converge for $|\psi| < 1$.

What we are seeking is a relation of the form

$$\Pi(\psi) = m\varpi(\psi), \tag{3.23}$$

with Π as in eq. (2.11) and with m a numerical matrix which in virtue of eqs. (3.22) and (3.7) is of the form

$$m = \begin{pmatrix} a & b & c & d \\ 0 & 0 & -1 & 0 \\ e & f & g & h \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

The periods z^1 and $\mathcal{G}_1$ are not as yet uniquely defined, since given any choice there is the freedom to make a second choice differing from the first by an $\mathrm{Sp}(2;\mathbb{Z}) = \mathrm{SL}(2;\mathbb{Z})$ transformation

$$\begin{pmatrix} \mathcal{G}_1 \\ z^1 \end{pmatrix} \to \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mathcal{G}_1 \\ z^1 \end{pmatrix}. \tag{3.24}$$

We can use this freedom to set $f = 0$ in m. We know also that the periods z^1 and $\mathcal{G}_1$ are free of logarithms at $\psi = 1$, which is the condition that the sums, weighted by the coefficients c_j, of the corresponding rows of m must vanish. Consider now how the basis ϖ changes under $\psi \to \alpha\psi$,

$$\varpi(\alpha\psi) = a\varpi(\psi),$$

with

$$a = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

It is this simple form for a that motivated the choice of basis ϖ. In terms of the symplectic basis we have

$$\Pi(\alpha\psi) = A\Pi(\psi), \qquad A = mam^{-1}.$$

The matrix A must be integral and symplectic. This turns out to be a stringent condition with a solution for m that is unique up to the $\mathrm{Sp}(2;\mathbb{Z})$ transformations (3.24),

$$m = \begin{pmatrix} -\frac{3}{5} & -\frac{1}{5} & \frac{21}{5} & \frac{8}{5} \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 8 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{3.25}$$

and we find

$$A = \begin{pmatrix} -9 & -3 & 5 & 3 \\ 0 & 1 & 0 & -1 \\ -20 & -5 & 11 & 5 \\ -15 & 5 & 8 & -4 \end{pmatrix}.$$

TABLE 1

The matrices associated with the transformation $\psi \to \alpha\psi$ and monodromy about $\psi = 1$ and $\psi = \infty$. The matrices associated with monodromy about $\psi = \alpha^k$ are $a^{-k}ta^k$ and $A^{-k}TA^k$

	ϖ	Π
$\psi \to \alpha\psi$	$a = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$A = \begin{pmatrix} -9 & -3 & 5 & 3 \\ 0 & 1 & 0 & -1 \\ -20 & -5 & 11 & 5 \\ -15 & 5 & 8 & -4 \end{pmatrix}$
monodromy about $\psi = 1$	$t = \begin{pmatrix} 1 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -4 & 1 \end{pmatrix}$	$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
monodromy about $\psi = \infty$	$t_\infty = \begin{pmatrix} -34 & -55 & -310 & 50 \\ 10 & 16 & 90 & -15 \\ 0 & 0 & 1 & 0 \\ -15 & -25 & -155 & 21 \end{pmatrix}$	$T_\infty = \begin{pmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{pmatrix}$

Given m we can return now to the monodromy of the bases about $\psi = 1$,

$$\varpi \to t\varpi, \qquad \Pi \to T\Pi,$$

where the matrices t and T may be found from (3.18) and (3.25). Given the transformations T and A, it is a simple matter to compute the monodromy of the bases about the other singularities in virtue of eqs. (2.12) and (2.13). We record these results in table 1. Referring to the table we see that the integer n in our previous expression for the monodromy matrix (2.10) is in fact unity.

4. The prepotential, metric and Yukawa coupling

Given the periods $(\mathscr{G}_a, z^b)$ it is now straightforward to construct the prepotential and the metric. The prepotential can be defined by

$$\mathscr{G} = \tfrac{1}{2} z^a \mathscr{G}_a .$$

Recall that the holomorphic three-form Ω has the variational property

$$\int_{\mathscr{W}} \Omega \wedge \frac{\partial \Omega}{\partial \psi} = 0,$$

and that this property requires the $\mathscr{G}_a$ to be the derivatives of the prepotential,

$$\mathscr{G}_a = \frac{\partial \mathscr{G}}{\partial z^a}, \qquad a = 1,2.$$

As a consistency check we can verify that this relation is satisfied. To this end note that, in virtue of the fact that the prepotential is homogeneous of degree two as a function of the z^a,

$$\mathscr{G}(z^1, z^2) = (z^2)^2 \mathscr{G}\left(\frac{z^1}{z^2}, 1\right).$$

A short calculation now reveals that

$$\frac{\partial \mathscr{G}}{\partial z^a} = \mathscr{G}_a - \epsilon_{ab} z^b \frac{W[z^c, \mathscr{G}_c]}{W[z^1, z^2]}$$

where ϵ_{ab} is the permutation symbol and

$$W[u,v] = u\frac{\mathrm{d}v}{\mathrm{d}\psi} - v\frac{\mathrm{d}u}{\mathrm{d}\psi}.$$

Thus for consistency the identity $W[z^c, \mathcal{G}_c] = 0$ must be satisfied. We have checked that it is indeed satisfied by the somewhat brutish method of expanding the periods as power series in ψ by means of eqs. (3.15) and (3.25) and checking that $W[z^c, \mathcal{G}_c]$ vanishes order by order. Actually, the vanishing of $W[z^c, \mathcal{G}_c]$ is a very natural condition as can be appreciated by noting that

$$\int_{\mathcal{W}} \Omega(\psi) \wedge \Omega(\psi') = z^c(\psi)\mathcal{G}_c(\psi') - z^c(\psi')\mathcal{G}_c(\psi). \tag{4.1}$$

By differentiating this expression with respect to ψ' and setting $\psi' = \psi$, we find that

$$W[z^c, \mathcal{G}_c] = \int_{\mathcal{W}} \Omega \wedge \frac{\partial \Omega}{\partial \psi},$$

and hence must vanish.

The Kähler potential K is given by the relations

$$\begin{aligned} e^{-K} &= i\left(\bar{z}^a \mathcal{G}_a - z^a \overline{\mathcal{G}}_a\right) \\ &= -\Pi^\dagger \Sigma \Pi \\ &= -i\varpi^\dagger \sigma \varpi, \end{aligned} \tag{4.2}$$

with

$$\Sigma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \sigma = -\frac{1}{5}\begin{pmatrix} 0 & 1 & 3 & 1 \\ -1 & 0 & 3 & 3 \\ -3 & -3 & 0 & 1 \\ -1 & -3 & -1 & 0 \end{pmatrix}.$$

The metric on the moduli space follows from (4.2). A three-dimensional plot of $g_{\psi\bar{\psi}}$ against ψ is presented in fig. 7. The cusp at $\psi = 1$ (there is of course only one cusp owing to the identification $\psi \approx \alpha\psi$) corresponds to the value of ψ for which $\mathcal{W}$ is a conifold. The metric is mildly singular at the conifold. It can be shown that $g_{\psi\bar{\psi}}$ is asymptotically proportional to $\log|\psi - 1|$, so the conifold is at a finite distance from the smooth manifolds in agreement with general results [17]. In order to compare the metric $g_{\psi\bar{\psi}}$ with the corresponding metric for the moduli space of $\mathbb{P}_4(5)$, it is of interest to compute the asymptotic form of the metric as $\psi \to \infty$. In appendix B series expansions, valid for $|\psi| > 1$, are derived for the ϖ_j,

$$\varpi_j(\psi) = \sum_{r=0}^{3} \log^r(5\psi) \sum_{n=0}^{\infty} b_{jrn} \frac{(5n)!}{(n!)^5 (5\psi)^{5n}}, \qquad |\psi| > 1.$$

In order to compute the leading terms of the metric we retain only the terms with

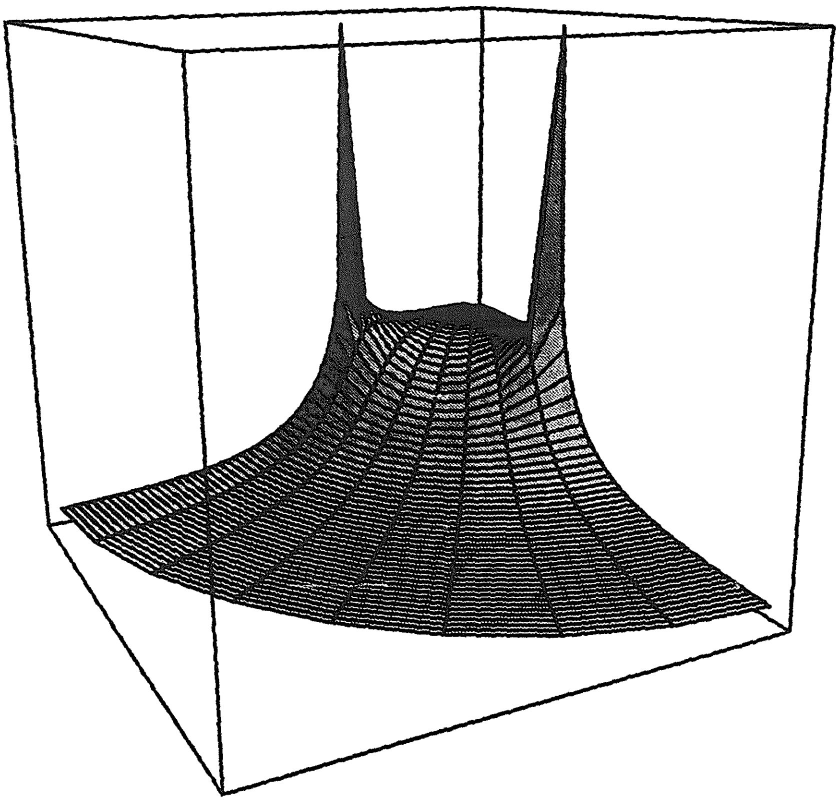

Fig. 7. A plot of the metric $g_{\psi\bar{\psi}}$ against ψ for ψ in the fundamental region $0 \leqslant \arg\psi < 2\pi/5$. The cusps correspond to the conifold at $\psi = 1$. As $\psi \to \infty$ the metric tends asymptotically to a metric of constant negative curvature.

$n = 0$ and define vectors of coefficients similar to ϖ,

$$b_r \stackrel{\text{def}}{=} -\left(\frac{2\pi i}{5}\right)^3 \begin{pmatrix} b_{2r0} \\ b_{1r0} \\ b_{0r0} \\ b_{4r0} \end{pmatrix},$$

so that

$$\varpi(\psi) \sim \sum_{r=0}^{3} b_r \log^r(5\psi).$$

Most of the terms

$$\sigma_{rs} \stackrel{\text{def}}{=} -ib_r^\dagger \sigma b_s$$

vanish, the only nonzero products being

$$\sigma_{30} = \sigma_{03} = \frac{4\pi^3}{75}, \qquad \sigma_{21} = \sigma_{12} = \frac{4\pi^3}{25}, \qquad \sigma_{00} = \tfrac{128}{625}\pi^3\zeta(3),$$

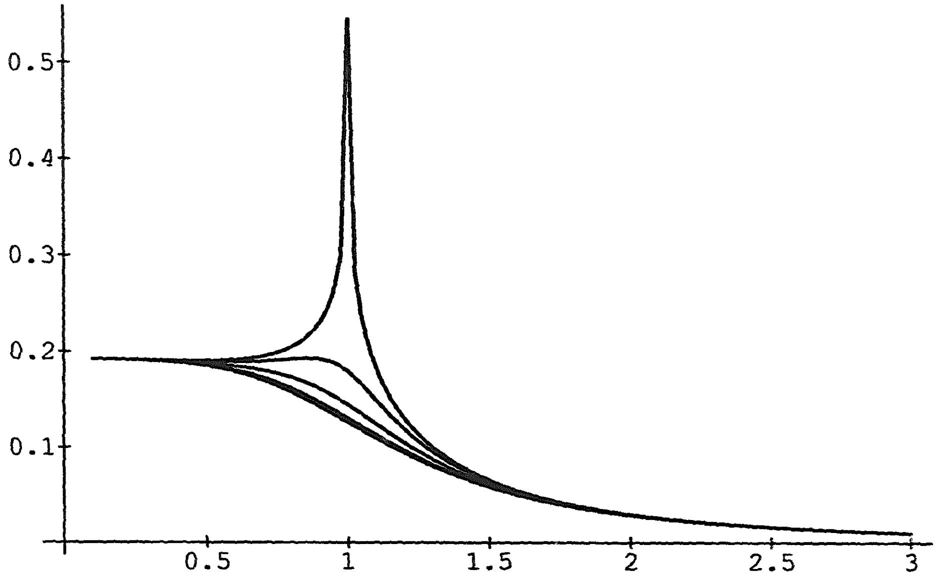

Fig. 8. A plot of the metric $g_{\psi\bar{\psi}}$ against $|\psi|$ for $\arg\psi = k\pi/20$, $k = (0, 1, 2, 3, 4)$. Note the cusp at $\psi = 1$ where the metric has a logarithmic singularity.

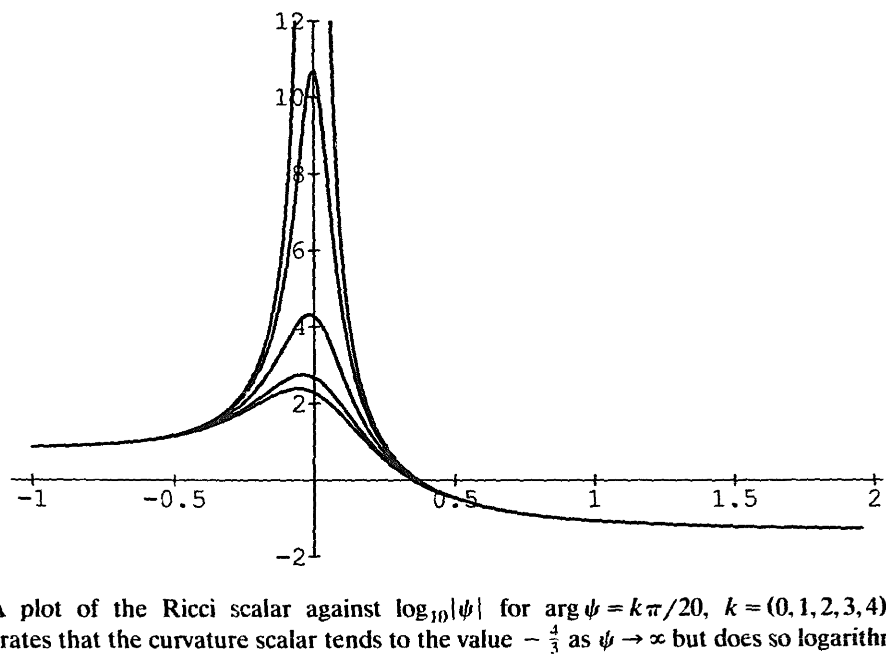

Fig. 9. A plot of the Ricci scalar against $\log_{10}|\psi|$ for $\arg\psi = k\pi/20$, $k = (0, 1, 2, 3, 4)$. The plot illustrates that the curvature scalar tends to the value $-\frac{4}{3}$ as $\psi \to \infty$ but does so logarithmically.

where ζ is the Riemann ζ-function. Thus

$$e^{-K} \sim \left(\frac{2\pi}{5}\right)^3 \left\{\tfrac{20}{3}\log^3|5\psi| + \tfrac{16}{5}\zeta(3)\right\},$$

and so we find the leading terms of the metric to be

$$g_{\psi\bar{\psi}} = \frac{3}{4|\psi|^2\log^2|5\psi|}\left\{1 - \frac{48\zeta(3)}{25\log^3|5\psi|} + \dots\right\}. \tag{4.3}$$

The leading term corresponds to a metric of uniform negative curvature. In fact, on setting

$$t \sim -\frac{5}{2\pi i}\log(5\psi),$$

we find that

$$ds^2 \sim \frac{3}{2}\frac{|dt|^2}{(t_2)^2},$$

where we have written t_2 for $\operatorname{Im} t$, so the large complex structure limit of the geometry coincides, as we shall see in the following section, with the large-radius limit of the moduli space for $\mathbb{P}_4(5)$. The subleading term in eq. (4.3) corresponds to a loop correction to the metric which we shall discuss further in sect. 5. Note finally that the singular manifold corresponding to $\psi = \infty$ is infinitely distant from the smooth manifolds.

4.1. THE YUKAWA COUPLING

In order to make contact with the Yukawa coupling we introduce a set of "wronskians",

$$W_k \stackrel{\text{def}}{=} z^a\left(\frac{d}{d\psi}\right)^k \mathcal{G}_a - \mathcal{G}_a\left(\frac{d}{d\psi}\right)^k z^a.$$

Then, from eq. (4.1), we find a relation between the Yukawa coupling and W_3,

$$\begin{aligned}\kappa_{\psi\psi\psi} &= \int \Omega \wedge \frac{\partial^3\Omega}{\partial\psi^3} \\ &= W_3.\end{aligned} \tag{4.4}$$

Now the form of W_3 follows straightforwardly from the differential equation (3.9) that governs the periods. Let us rewrite the equation in the form

$$\left\{\left(\frac{d}{d\psi}\right)^4 + \sum_{k=0}^{3} C_k(\psi)\left(\frac{d}{d\psi}\right)^k\right\}\Pi = 0.$$

From this relation it is immediate that

$$W_4 + \sum_{k=0}^{3} C_k W_k = 0. \tag{4.5}$$

Note now that

$$W_0 = W_1 = W_2 = 0.$$

The vanishing of W_0 is trivial. W_2 is the derivative of W_1 and we have verified that W_1 vanishes. Note also the elementary identity

$$W_4 - 2W_3' + W_2'' = 0.$$

Putting these results together we find

$$W_3' + \tfrac{1}{2}C_3 W_3 = 0,$$

and hence

$$\kappa_{\psi\psi\psi} = W_3 = \left(\frac{2\pi i}{5}\right)^3 \frac{5\psi^2}{1-\psi^5}, \tag{4.6}$$

the constant being determined by the computation of the specific form of W_3 as $\psi \to 0$. We could also have arrived at this same result by using the methods of ref. [10].

Some comments are in order here concerning the gauge dependence of the Yukawa couplings (for a full account in the spirit of the present work see refs. [20, 21]). It was noted previously that the holomorphic three-form is in reality undefined up to multiplication by a holomorphic function of ψ,

$$\Omega \to f\Omega,$$

and we refer to such a replacement as a gauge transformation. A gauge transformation induces a transformation of the period vector, $\Pi \to f\Pi$, and from eq. (4.4) it follows that the coupling is gauge dependent,

$$\kappa_{\psi\psi\psi} \to f^2 \kappa_{\psi\psi\psi};$$

so is the Kähler potential which transforms according to the rule

$$\mathrm{e}^{-K} \to |f|^2\, \mathrm{e}^{-K}.$$

The gauge choice is arbitrary and a physical quantity such as a decay rate cannot depend on the gauge. It must therefore be the case that the coupling enters into physical quantities only through the invariant combination

$$y_{\mathrm{inv}} \stackrel{\mathrm{def}}{=} g^{-3/2}\, \mathrm{e}^{K} |\kappa|,$$

the factor of $g^{-3/2}$ being included to remove the coordinate dependence of $|\kappa|$ (since the effect of a gauge transformation on the Kähler potential is $K \to K - \log f - \log \bar{f}$ the metric is gauge invariant).

Roughly speaking the mechanism that causes κ to be replaced by the invariant coupling is the following. The low-energy lagrangian contains terms of the form

$$g\,\mathrm{e}^{-K}\bar{\chi}\not{D}\chi + \tfrac{1}{2}g|\partial\phi|^2 + (\kappa\chi\chi\phi + \text{conjugate}).$$

With a suitable choice of covariant derivative D, this is invariant under

$$\kappa \to f^2\kappa, \qquad \mathrm{e}^{-K} \to |f|^2\,\mathrm{e}^{-K},$$

if also

$$\chi \to f^{-1}\chi,$$

and ϕ is invariant. We can normalize the kinetic terms by defining

$$\tilde{\chi} = g^{1/2}\,\mathrm{e}^{-K/2}\chi\,\mathrm{e}^{i\theta/2} \qquad \tilde{\phi} = g^{1/2}\phi,$$

where $\theta = \arg\kappa$. Then the low-energy terms become

$$\bar{\tilde{\chi}}\not{D}\tilde{\chi} + \tfrac{1}{2}|\partial\tilde{\phi}|^2 + g^{-3/2}\,\mathrm{e}^{K}|\kappa|\left(\tilde{\chi}\tilde{\chi}\tilde{\phi} + \text{conjugate}\right).$$

We summarize the limiting forms of the metric, curvature and invariant Yukawa couplings in table 2. The reader will recognize the value obtained for the invariant coupling at $\psi = 0$ [7] as being the value corresponding to the Gepner model 3^5 [25]. This agreement is an important check on the mirror hypothesis. The $\psi = \infty$ row of table 2 gives the "bare" value of y_{inv} as being $2/\sqrt{3}$. In ref. [7] this value is incorrectly stated to be $1/\sqrt{3}$.

Finally we note that for a one-dimensional manifold that is special Kähler, the Ricci scalar is related to the invariant coupling by

$$R + 4 = 2y_{\text{inv}}^2, \tag{4.7}$$

TABLE 2

The asymptotic/limiting forms of the metric, the scalar curvature and the invariant Yukawa coupling for $\psi = 0, 1, \infty$. In the middle row a is a positive constant and $r = |\psi - 1|$.

ψ	$g_{\psi\bar{\psi}}$	R	$g^{-3/2}\,\mathrm{e}^{K}|\kappa|$
0	$25\dfrac{\Gamma^5(\frac{4}{5})\Gamma^5(\frac{2}{5})}{\Gamma^5(\frac{1}{5})\Gamma^5(\frac{3}{5})}$	$2\dfrac{\Gamma^{15}(\frac{3}{5})\Gamma^5(\frac{1}{5})}{\Gamma^{15}(\frac{2}{5})\Gamma^5(\frac{4}{5})} - 4$	$\dfrac{\Gamma^{15/2}(\frac{3}{5})\Gamma^{5/2}(\frac{1}{5})}{\Gamma^{15/2}(\frac{2}{5})\Gamma^{5/2}(\frac{4}{5})}$
1	$-a^2\log r$	$\dfrac{1}{2a^2r^2[-\log r]^3}$	$\dfrac{1}{2ar[-\log r]^{3/2}}$
∞	$\dfrac{3}{4|\psi|^2\log^2|\psi|}$	$-\dfrac{4}{3}$	$\dfrac{2}{\sqrt{3}}$

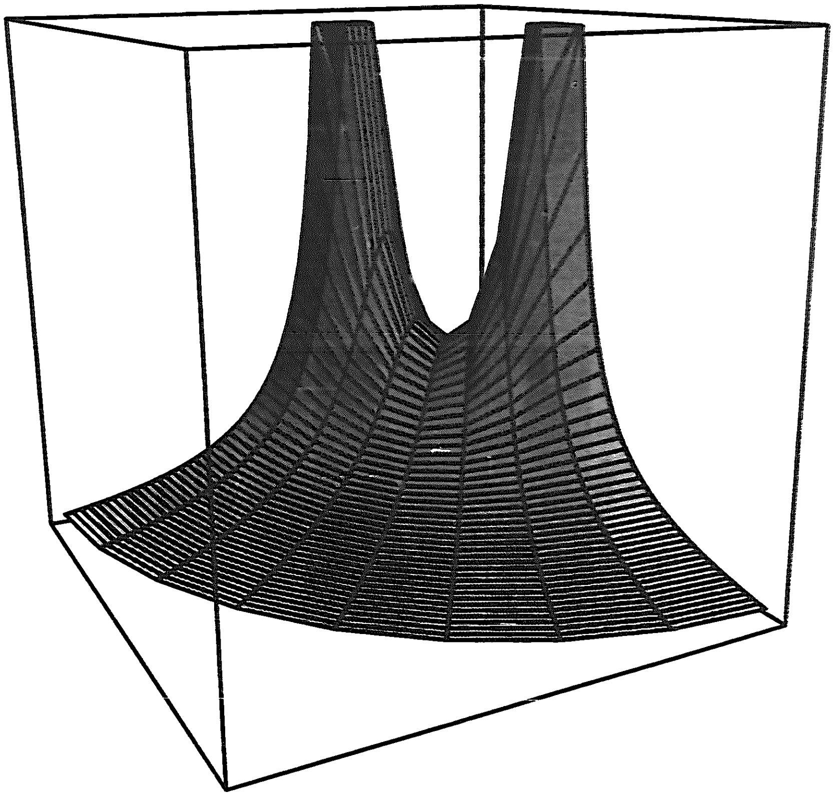

Fig. 10. A plot of the Ricci scalar R against ψ for ψ in the fundamental region. The curvature tends to infinity at $\psi = 1$. As $\psi \to \infty$ the curvature tends asymptotically to the constant value $-\frac{4}{3}$. In virtue of eq. (4.7) this figure has essentially the same form as a plot of the invariant Yukawa coupling.

and we present a three-dimensional plot of the Ricci scalar in fig. 10. A plot of y_{inv} has the same form. The "bare" value of y_{inv} is $2/\sqrt{3}$. The quantum corrections cause y_{inv} to differ from this constant value and the correction becomes infinite at the conifold.

5. $\mathbb{P}_4(5)$, the mirror map and quantum corrections

Recall [20] that prior to receiving quantum corrections, the prepotential is given in terms of $b_{11}+1$ homogeneous coordinates w^j by the expression

$$\mathcal{F}(w) = -\frac{1}{3!}\frac{\kappa_{ABC}w^A w^B w^C}{w^0}, \qquad A = 1,\ldots,b_{11},$$

where the

$$\kappa_{ABC} = \int_{\mathcal{M}} e_A \wedge e_B \wedge e_C$$

are the intersection numbers of a basis for $H^2(\mathcal{M},\mathbb{Z})$. For the present case $b_{11}=1$, and denoting the generator of $H^2(\mathbb{P}_4(5),\mathbb{Z})$ by e we have

$$\int_{\mathcal{M}} e\wedge e\wedge e = 5. \tag{5.1}$$

Thus, using coordinates (w^1,w^2) rather than (w^0,w^1), we write

$$\mathcal{F} = -\frac{5}{6}\frac{\left(w^1\right)^3}{w^2} = -\tfrac{5}{6}\left(w^2\right)^2 t^3, \tag{5.2}$$

the latter form being written in terms of an affine coordinate $t \stackrel{\text{def}}{=} w^1/w^2$. It is simplest to set $w^2=1$, after differentiation, then the Kähler potential is given by

$$K = -\log\left\{ i\left(\overline{w}^j \frac{\partial\mathcal{F}}{\partial w^j} - w^j \frac{\partial\overline{\mathcal{F}}}{\partial\overline{w}^j}\right)\right\}$$

$$= -\log\left(\tfrac{20}{3}(t_2)^3\right). \tag{5.3}$$

From this we find that the metric and Ricci tensor on the parameter space are given by

$$g_{t\bar{t}} = \frac{3}{4(t_2)^2}, \qquad R_{t\bar{t}} = -\tfrac{2}{3}g_{t\bar{t}}.$$

We summarize these results in table 3, which should be compared with the $\psi=\infty$ row of table 2.

We have seen that the large complex structure limit of the invariant Yukawa coupling and the geometry of $\mathcal{W}$ agree with the bare values of the corresponding quantities for $\mathcal{M}$. We wish to examine the quantum corrections to the bare quantities by comparing corresponding quantities on $\mathcal{W}$ and $\mathcal{M}$.

A detailed comparison requires the explicit form of the mirror map $t\to\psi(t)$ between the two parameter spaces. In order to do this it turns out that we need to understand the quadratic terms in the prepotential and these in turn involve the

TABLE 3
The metric, the scalar curvature and the invariant Yukawa coupling as functions of t

t	$g_{t\bar{t}}$	R	$g^{-3/2}\,e^{K}\lvert\kappa\rvert$
all t	$\dfrac{3}{4t_2^2}$	$-\dfrac{4}{3}$	$\dfrac{2}{\sqrt{3}}$

loop corrections, so we shall first discuss the loop corrections. Perhaps surprisingly we shall see that there is a single loop correction to the bare prepotential.

5.1. THE LOOP TERM

We have of course a nonrenormalization theorem [5], to the effect that there are no sigma model loop corrections to the superpotential or equivalently, to the couplings $\partial^3\mathscr{F}/\partial w^a \partial w^b \partial w^c$. The prepotential is homogeneous of degree two, so it is determined by its third derivative up to a quadratic term of the form

$$\Delta\mathscr{F} = \tfrac{1}{2} w^{\mathrm{T}} \mathscr{Z} w\,,$$

with $\mathscr{Z}$ a symmetric constant matrix. We can decompose $\mathscr{Z}$ into its real and imaginary parts,

$$\mathscr{Z} = \mathscr{X} + i\mathscr{Y}\,.$$

The real and imaginary parts affect the prepotential differently. From eq. (5.3), which may be written in the form

$$\mathrm{e}^{-K} = 2w^{\dagger}(\operatorname{Im}\mathbb{F})w\,,$$

with $\mathbb{F}$ the matrix of second derivatives of $\mathscr{F}$, we see that the real part of $\mathscr{Z}$ contributes neither to the Yukawa couplings nor to the metric of the moduli space. Such a term is of no consequence and could be discarded. Alternatively, we can absorb it into the bare part, $\mathscr{F}_0$, of the prepotential by extending the range of the indices on the intersection numbers κ_{ABC} to include the value zero, so that

$$\mathscr{F}_0 = -\frac{1}{3!}\frac{\kappa_{ABC}w^A w^B w^C}{w^0} + \tfrac{1}{2}\mathscr{X}_{ab}w^a w^b = -\frac{1}{3!}\frac{\kappa_{abc}w^a w^b w^c}{w^0}\,,$$

$$a,b = 0,1,\ldots,b_{11}\,.$$

The imaginary part of $\mathscr{Z}$ does affect the metric and since the contribution of $\mathscr{Y}$ to the prepotential,

$$\mathscr{F}_{\text{loop}} \stackrel{\text{def}}{=} i w^{\mathrm{T}}\mathscr{Y} w\,,$$

contains powers of the radius (recall that for the case of $\mathbb{P}_4(5)$, $t = w^1/w^2$ and the imaginary part of t is the square of the radius), it is precisely the loop correction to $\mathscr{F}_0$.

We shall now argue that $\mathscr{F}_{\text{loop}}$ contains a four-loop correction and that this is the only loop correction to the prepotential*. In fact $\mathscr{F}_{\text{loop}}$ is closely related to the

* The discussion presented here grew out of animated conversations with A. Strominger.

four-loop term found by Grisaru et al. [14] in their investigation of the β-function for type-(2, 2) supersymmetric σ-models. The loop corrections arise because there are loop corrections to the sigma model. When the heavy modes are integrated out of the sigma model, part of the loop term survives into the low-energy theory. When we refer to a four-loop term, we mean a four-loop contribution to the sigma model. This corresponds to a *third*-order correction to the prepotential.

The particular case of $\mathbb{P}_4(5)$ corresponds to $b_{11} = 1$ but it is simpler and more general to leave b_{11} arbitrary for the present. The large-radius limit is the limit $w^0 \to 0$ for fixed w^A (this is the limit $t \to \infty$ for $\mathbb{P}_4(5)$) and the loop counting is that $1/w^0$ corresponds to one loop and each additional power of w^0 corresponds to an additional loop. Thus the term $iw^{\mathrm{T}}\mathscr{Y}w$ contains, in principle, two-loop, three-loop and four-loop terms. These terms must be constructed from local polynomials in the curvature of $\mathscr{M}$. These terms have to be universal, that is independent of any structure, such as complex structure or the dimension of the space, that is particular to $\mathscr{M}$. Furthermore, the loop contributions must vanish for hyper-Kähler manifolds. These conditions, which are clearly very restrictive, were investigated in relation to the counterterms that can arise for the β-functions for type-(2, 2) supersymmetric σ-models [26]. It is known that there are no curvature polynomials with the requisite properties that could correspond to two or three loops and that there is a single such quantity at four loops,

$$S \stackrel{\text{def}}{=} \mathsf{R}_{ij}{}^{kl}\mathsf{R}_{kl}{}^{mn}\mathsf{R}_{mn}{}^{ij} - 2\mathsf{R}_{i\ j}^{\ k\ l}\mathsf{R}_{k\ l}^{\ m\ n}\mathsf{R}_{m\ n}^{\ i\ j},$$

where R_{ijkl} is the curvature tensor of $\mathscr{M}$. A further remarkable property of S for Calabi–Yau manifolds in six dimensions is that S is proportional to the Euler integrand. It can be shown that

$$Sg^{1/2}\,\mathrm{d}^6x = 12(2\pi)^3 c_3,$$

where c_3 is the third Chern class.

Consider now the possible structure of the components $\mathscr{Y}_{ab}$. The component $\mathscr{Y}_{00}$ can be of the form

$$\mathscr{Y}_{00} = a\int_{\mathscr{M}} \mathrm{d}^6x\, g^{1/2} S = 12(2\pi)^3 a\chi,$$

with a a numerical coefficient. The other components of $\mathscr{Y}$ all vanish. The terms $\mathscr{Y}_{AB}w^Aw^B$ and $\mathscr{Y}_{0A}w^0w^A$ are respectively one- and two-loop terms for which, as already mentioned, suitable curvature polynomials do not exist.

We therefore know the form of $\mathscr{F}_{\text{loop}}$ up to a numerical coefficient independent of $\mathscr{M}$. One way to fix the coefficient would be to actually evaluate a four-loop graph. It is simpler however to compare with the corrected prepotential for $\mathbb{P}_4(5)$

which we do in eq. (5.10) below. In this way we find that

$$\mathscr{F}_{\text{loop}} = \frac{i\zeta(3)}{2(2\pi)^3}\chi \,. \tag{5.4}$$

For the case of $\mathbb{P}_4(5)$ it is this contribution to the prepotential which produces the subleading term in the metric in eq. (4.3) and is responsible for the fact that, for large ψ, the Ricci scalar of the moduli space differs from its limiting value by inverse powers of $\log\psi$ as is evident from fig. 9.

5.2. THE MIRROR MAP

We find the relation between the coordinates ψ and t by relating the period vector Π to the corresponding vector Π for $\mathbb{P}_4(5)$. The idea is that we expect the prepotential $\mathscr{G}$ calculated in sect. 4 to be the fully corrected form of the bare prepotential $\mathscr{F}_0$ given by (5.2). The complication is that the form of the prepotential depends on the choice of symplectic basis. So we must first find the relation between the symplectic bases for $\mathscr{W}$ and $\mathscr{M}$. We do this by relating Π and Π for large radius and large complex structure.

The leading behaviour of Π is derived from the leading behaviour of the prepotential $\mathscr{F}$, which we take to be

$$\mathscr{F}_{\text{pert}} \overset{\text{def}}{=} \mathscr{F}_0 + \mathscr{F}_{\text{loop}} = -\frac{5}{6}\frac{(w^1)^3}{w^2} + \tfrac{1}{2}a(w^1)^2 + bw^1w^2 + \tfrac{1}{2}c(w^2)^2 .$$

The first term is as in eq. (5.2) and we have allowed for quadratic terms in line with our discussion above. From $\mathscr{F}_{\text{pert}}$ we obtain the leading behaviour of the period vector,

$$\Pi_{\text{pert}} \overset{\text{def}}{=} \begin{pmatrix} \dfrac{\partial \mathscr{F}_{\text{pert}}}{\partial w^1} \\ \dfrac{\partial \mathscr{F}_{\text{pert}}}{\partial w^2} \\ w^1 \\ w^2 \end{pmatrix} = w^2 \begin{pmatrix} -\frac{5}{2}t^2 + at + b \\ \frac{5}{6}t^3 + bt + c \\ t \\ 1 \end{pmatrix} . \tag{5.5}$$

We have the elementary fact that Π_{pert} is cubic in t. On the other hand we have, as $\psi \to \infty$,

$$\frac{\Pi}{\mathscr{G}_2} \sim t^3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5/6 \end{pmatrix} + t^2 \begin{pmatrix} 5/2 \\ 0 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 11/2 \\ 0 \\ 10 \\ -25/12 \end{pmatrix} + \begin{pmatrix} -25/12 \\ 1 \\ -25/6 \\ 0 \end{pmatrix} + \frac{25i}{\pi^3}\zeta(3)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \tag{5.6}$$

where

$$\mathscr{G}_2 \sim \left(\frac{2\pi i}{5}\right)^3,$$

with asymptotic equality meaning equality up to terms that are $\mathscr{O}(\psi^{-5}\log^3(5\psi))$. We have also taken

$$t \sim -\frac{5}{2\pi i}\log(5\psi)$$

as before and we again mean equality up to terms that involve ψ^{-5} multiplied by logarithms. Because of the relation between t and ψ the neglected terms are $\mathscr{O}(t^k \mathrm{e}^{2\pi i t})$, and with a slight abuse of language we shall refer to such terms as being exponentially small in the large-radius ($\mathrm{Im}\, t \to \infty$) limit. We allow for a symplectic transformation N and set

$$\Pi_{\text{pert}} \sim N\Pi. \tag{5.7}$$

Since both (5.5) and (5.6) are cubics we can solve for N,

$$N = \begin{pmatrix} -1+2a' & b' & -a' & 0 \\ 2b' & c' & -b' & -1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where

$$a = -\tfrac{11}{2} + a', \qquad b = \tfrac{25}{12} + b', \qquad c = -\frac{25i}{\pi}\zeta(3) + c'.$$

If we take $a', b', c' \in \mathbb{Z}$ then $N \in \mathrm{Sp}(4;\mathbb{Z})$ for all a', b', c'. The fact that Π_{pert} and Π are related by a symplectic matrix is a consequence of mirror symmetry. The fact that they may be related by an integral symplectic matrix is the observation already made by Aspinwall and Lütken [27] that the mirror symmetry identifies the two lattices

$$\Lambda = \mathrm{H}^3(\mathscr{W}, \mathbb{Z}), \qquad \mathrm{V} = \bigoplus_{i=0}^{3} \mathrm{H}^{2i}(\mathscr{M}, \mathbb{Z}).$$

We shall here take a', b', c' to vanish, other choices corresponding merely to a further $\mathrm{Sp}(4,\mathbb{Z})$ transformation. So the final form for N is

$$N = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

With sufficient prescience we could of course have chosen a basis for Π so that N would have turned out to be the identity.

With N in hand we may write down the quantum version of Π, with the gauge choice $w^2 = \mathscr{G}_2$,

$$\Pi = N\Pi, \qquad w^2 = \mathscr{G}_2 .$$

The gauge-invariant form of this relation is

$$\Pi = \frac{w^2}{\mathscr{G}_2} N\Pi . \tag{5.8}$$

Our aim here is to compare the quantum prepotential $\mathscr{F}$ that derives from Π with that derived from $\mathscr{F}_0$. We set

$$\Pi = \begin{pmatrix} \mathscr{F}_1 \\ \mathscr{F}_2 \\ w^1 \\ w^2 \end{pmatrix},$$

and referring back to eqs. (3.23) and (3.25), recalling that the latter expression for Π is in the gauge $\mathscr{G}_2 = (2\pi i/5)^3 \varpi_0$, we find the following expression for t:

$$\begin{aligned} t = \frac{w^1}{w^2} &= \frac{2(\varpi_1 - \varpi_0) + \varpi_2 - \varpi_4}{5\varpi_0} \\ &= -\frac{5}{2\pi i}\left\{ \log(5\psi) - \frac{1}{\varpi_0(\psi)} \sum_{m=1}^{\infty} \frac{(5m)!}{(m!)^5 (5\psi)^{5m}} \left[\Psi(1+5m) - \Psi(1+m) \right] \right\} . \end{aligned} \tag{5.9}$$

This equation gives the mirror map. We have introduced the coordinate t partly to make contact with previous work but primarily because $2k\pi i t$ is the value of the action evaluated on a rational curve of degree k (that is on an instanton of degree k). The coordinate t transforms in a complicated way under the generators $\mathscr{A}$ and $\mathscr{T}$ of modular transformations, although under $(\mathscr{T}\mathscr{A})^{-1}$ we have the simple rule

$$(\mathscr{T}\mathscr{A})^{-1}: \quad t \to t+1 .$$

We also have

$$\mathscr{F} = \tfrac{1}{2} w^a \mathscr{F}_a = (w^2)^2 \left\{ -\tfrac{5}{6} t^3 - \tfrac{11}{4} t^2 + \tfrac{25}{12} t - \frac{25 i}{2\pi^3} \zeta(3) + \text{exponentially small} \right\}, \tag{5.10}$$

where the last expression follows on inverting eq. (5.9) to give $\psi = \psi(t)$. Note the structure of $\mathscr{F}$. First there are terms that are cubic, quadratic and linear in t that have real coefficients. These correspond to the bare prepotential $\mathscr{F}_0$. Then there is the term independent of t, which we identify as the loop term and which fixes the coefficient in eq. (5.4). Lastly there are exponentially small terms which may be thought of as instanton corrections.

5.3. THE SUM OVER INSTANTONS

We wish to examine these exponentially small terms. The idea is to examine the difference between $\mathscr{G}$ and $\mathscr{F}_0$. However, as mentioned above, one must also allow for the effect of the symplectic transformation N. It is simpler to consider first the instanton contribution to the Yukawa coupling, which is of interest in its own right, the corresponding contributions to $\mathscr{F}$ can then be obtained by integration. We have

$$\kappa_{ttt} = \left(\frac{w^2}{\mathscr{G}_2}\right)^2 \kappa_{\psi\psi\psi}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^3, \tag{5.11}$$

where $\kappa_{\psi\psi\psi}$ is given by eq. (4.6) and the prefactor expresses the gauge freedom. If we choose $w^2 = 1$ and recall that $\kappa_{\psi\psi\psi}$ was derived in the gauge $\mathscr{G}_2 = (2\pi i/5)^3 \varpi_0$, we find

$$\kappa_{ttt} = 5 + 2875\,\mathrm{e}^{2\pi i t} + 4876875\,\mathrm{e}^{4\pi i t} + \dots, \qquad w^2 = 1. \tag{5.12}$$

To understand the structure of this sum consider the contribution of a rational curve $\mathscr{L}$ of degree k to the coupling. A rational curve of degree k is a $\mathbb{P}_1$ embedded by equations of degree k. Equivalently it is characterized by the fact that

$$\int_{\mathscr{L}} e = k,$$

where e is the generator of $\mathrm{H}^2(\mathbb{P}_4(5), \mathbb{Z})$. The contribution of $\mathscr{L}$ to the coupling is [7]

$$\exp\left(2\pi i t \int_{\mathscr{L}} e\right)\left(\int_{\mathscr{L}} e\right)^3 = k^3\,\mathrm{e}^{2\pi i k t}.$$

It is necessary to consider also multiple covers of the rational curves. For example, at degree 2 there are embeddings $\mathbb{P}_1 \hookrightarrow \mathbb{P}_4(5)$ given by

(i): $(u, v) \to (u^2, v^2, uv, 0, 0)$,

(ii): $(u, v) \to (u^2, v^2, 0, 0, 0)$.

These are given as embeddings $\mathbb{P}_1 \hookrightarrow \mathbb{P}_4$, but for suitable quintics the curves lie in the quintic hypersurface. The image in case (i) is a rational curve of degree 2 while case (ii) is a double cover of the line $(u,v,0,0,0)$ which is a rational curve of degree 1. Note that the two cases are intrinsically different, we cannot by means of a coordinate transformation transform case (ii) to case (i). More generally, an m-fold multiple cover of a rational curve of degree k is an embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_4(5)$ such that the homogeneous coordinates of the $\mathbb{P}_4$ are polynomials of degree k in quantities (U,V) that are themselves polynomials of degree m in the homogeneous coordinates of the $\mathbb{P}_1$. One difference between the multiple covers and the single covers is that it appears that the single covers have no parameters (it is proved in ref. [28] that the single covers have no parameters for $k \leqslant 7$) while m-fold covers with $m > 1$ *do* have parameters. Since the quantities U and V are polynomials of degree m in the coordinates (u,v) of the $\mathbb{P}_1$ they each contain $m+1$ parameters. Taking into account the three degrees of freedom in a reparametrization

$$(u,v) \to (au+bv, cu+dv), \qquad ad-bc=1,$$

of the $\mathbb{P}_1$ and the freedom to multiply U and V by a common scale, we find

$$2(m+1)-3-1=2(m-1)$$

parameters.

We shall assume that an m-fold cover contributes an amount $k^3\,\mathrm{e}^{2\pi i m k t}$ to the coupling. Since

$$\int_{m\mathscr{L}} e = mk\,,$$

this amounts to the assumption that m-fold covers have an associated prefactor $1/m^3$ (otherwise the contribution would be $(mk)^3\,\mathrm{e}^{2\pi i m k t}$). We believe that this prefactor derives from the zero modes associated with the parameters that the m-fold covers enjoy, but we have not performed the computation. The reason for assuming this specific form for the prefactor will become apparent as we proceed. With this understanding we find that the contribution to the coupling of a rational curve $\mathscr{L}$ of degree k together with all its multiple covers is

$$k^3 \sum_{m=1}^{\infty} \mathrm{e}^{2\pi i k m t} = \frac{k^3\,\mathrm{e}^{2\pi i k t}}{1-\mathrm{e}^{2\pi i k t}}\,.$$

Let n_k be the number of rational curves of degree k, then we have the following expression for the coupling as a sum over instantons:

$$\kappa_{ttt} = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3\,\mathrm{e}^{2\pi i k t}}{1-\mathrm{e}^{2\pi i k t}} = 5 + n_1\,\mathrm{e}^{2\pi i t} + \left(2^3 n_2 + n_1\right)\mathrm{e}^{4\pi i t} + \dots \tag{5.13}$$

It is gratifying that we find that $n_1 = 2875$ which is indeed the number of lines [29]* (rational curves of degree one) and $n_2 = 609\,250$ which is known to be the

*This number is misleadingly given as 375 in ref. [6]. This is due to the fact that the calculation is performed for a special class of quintics and no account taken of multiplicity. If the counting is done for a generic quintic then the result is 2875.

TABLE 4
The numbers of rational curves of degree k for $1 \leqslant k \leqslant 10$

k	n_k
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 58888 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

number of conics [28] (rational curves of degree two). Clemens has shown [30] that $n_k \neq 0$ for infinitely many k and has conjectured that $n_k \neq 0$ for all k, but it seems that the direct calculation of these numbers becomes difficult beyond $k = 2$ (see also ref. [28]). It is however straightforward to develop the series (5.12) to more terms and to find the n_k by comparison with (5.13). We present the first few n_k in table 4. These numbers provide compelling evidence that our assumption about the form of the prefactor is in fact correct. The evidence is not so much that we obtain in this way the correct values for n_1 and n_2, but rather that the coefficients in eq. (5.12) have remarkable divisibility properties. For example asserting that the second coefficient 4,876,875 is of the form $2^3 n_2 + n_1$ requires that the result of subtracting n_1 from the coefficient yields an integer that is divisible by 2^3. Similarly, the result of subtracting n_1 from the third coefficient must yield an integer divisible by 3^3. These conditions become increasingly intricate for large k. It is therefore remarkable that the n_k calculated in this way turn out to be integers.

The values for the n_k shown in the table are particular to $\mathbb{P}_4(5)$, however we can abstract from eq. (5.13) a form for the mirror map which we conjecture to be of general validity,

$$\mathscr{W}_w = \mathscr{M}_w + \sum_{\mathscr{L} \in \mathscr{M}} \frac{e^{2\pi i \mathscr{L}[w]}}{1 - e^{2\pi i \mathscr{L}[w]}} \mathscr{L}^3 , \tag{5.14}$$

where we regard the complex structure of $\mathscr{W}$ as being parametrized by the complex Kähler form $w = B + iJ$ of $\mathscr{M}$, and

$$\mathscr{L}[w] = \int_{\mathscr{L}} w .$$

Eq. (5.14) embodies the moral of the present work. There is a "bare manifold" $\mathscr{M}$ and a "quantum manifold" $\mathscr{W}$ and the quantum manifold is the bare manifold together with its rational curves.

5.4. THE NUMBER OF RATIONAL CURVES OF LARGE DEGREE

It is immediately apparent from table 4 that the n_k grow very rapidly with k. It is of interest to observe that the distribution of the n_k for large k is governed by the conifold at $\psi = 1$. Consider again the series (5.12) which gives the Yukawa coupling as a power series in $e^{2\pi i t}$. The radius of convergence of the series is determined by the singularity of the Yukawa coupling and the coupling is singular only when $\psi^5 = 1$, so we conclude that the series (5.12) converges for

$$\operatorname{Im} t > \operatorname{Im} t(1)\,.$$

From this it follows that n_k cannot grow faster than $e^{-2\pi i k t(1)}$ for large k. However, we can do better and find the asymptotic form of n_k from the singularity of the coupling.

First we need to know the behaviour of $t(\psi)$ as $\psi \to 1$. From eq. (5.9) we find

$$t(\psi) - t(1) \sim -\frac{5^{3/2}}{4\pi^2}\frac{it_2(1)}{\varpi_0(1)}(\psi - 1)\log(\psi - 1)\,.$$

Together with eq. (5.11) this enables us to find the leading behaviour of the singularity of the coupling as $\psi \to 1$,

$$\kappa_{ttt} \sim \frac{(2\pi)^3}{5^{3/2}}\frac{\varpi_0^3(1)}{t_2^3(1)}\frac{1}{(\psi - 1)[-\log(\psi - 1)]^3}\,. \tag{5.15}$$

On the other hand the severity of the singularity at $\psi \to 1$ must be dictated by the asymptotic behaviour of the n_k for large k. If we set

$$n_k \sim B k^{\rho - 3} \log^\sigma k \, e^{2\pi k t_2(1)}\,,$$

then we have

$$\begin{aligned}
\kappa_{ttt} &\sim B \sum^{\infty} k^\rho \log^\sigma k \, e^{-2\pi k (t_2(\psi) - t_2(1))} \\
&\sim B \int_0^\infty \mathrm{d}k\, k^\rho \log^\sigma k \, e^{-kx}\,, \qquad x \stackrel{\text{def}}{=} 2\pi\big(t_2(\psi) - t_2(1)\big)\,, \\
&\sim B \frac{\Gamma(1+\rho)}{x^{1+\rho}}(-\log x)^\sigma\,.
\end{aligned} \tag{5.16}$$

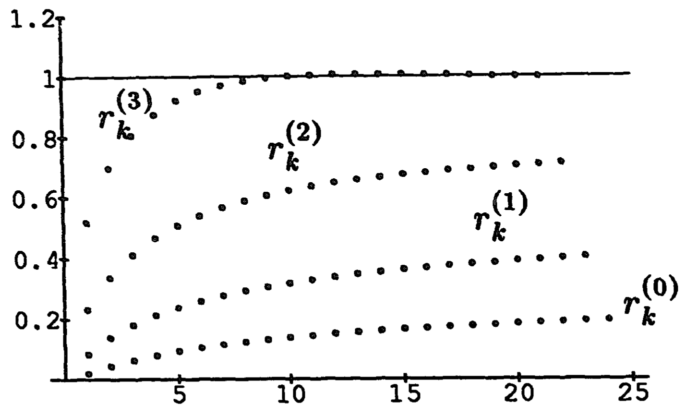

Fig. 11. A plot of the series of ratios $\{r_k^{(0)}\}$, for $2 \leqslant k \leqslant 25$, together with its first three Richardson transforms. The third transform converges rapidly to unity.

By comparing (5.16) with (5.15) we see that $\rho = 0$, $\sigma = -2$ and we find also a value for B. Thus we have

$$n_k \sim \left(\frac{2\pi \varpi_0(1)}{t_2(1)} \right)^2 \frac{e^{2\pi k t_2(1)}}{k^3 \log^2 k},$$

$$t_2(1) = 1.2081280770779186381 92\ldots, \qquad \varpi_0(1) = 1.0707258684301558005 71\ldots, \tag{5.17}$$

the next-order terms being smaller by inverse powers of $\log k$. Note that, as anticipated, the asymptotic form just refers to quantities evaluated at $\psi = 1$.

As a check we have plotted the values of

$$r_k^{(0)} = \frac{n_k}{\nu_k}, \tag{5.18}$$

where ν_k is the expression on the right-hand side of (5.17) in fig. 11 for k in the range $2 \leqslant k \leqslant 25$. The series $\{r_k^{(0)}\}$ converges very slowly owing to the fact that the subleading terms in the asymptotic expansion fall off as inverse powers of $\log k$. To speed the convergence we apply a variant of a Richardson transformation [31] to the series, the aim being to eliminate the effect of the subleading terms. We set

$$r_k^{(m)} = \frac{r_{k+1}^{(m-1)} \log(k+1) - r_k^{(m-1)} \log k}{\log((k+1)/k)},$$

and we record the values of the series $\{r_k^{(m)}\}$ in fig. 11 for $1 \leqslant m \leqslant 3$. We see that the third transform converges rapidly to unity.

5.5. SOME FURTHER REMARKS ON THE MODULAR GROUP

We saw previously that the modular group Γ acts most simply on a modular parameter γ. A puzzling fact is that γ does not appear to be the quantity that is of most direct physical interest. For example, the coordinate that arises naturally in the instanton sum is the coordinate t which is related to the complex Kähler form by the relation $B + iJ = te$. The coordinate t is, as we have seen, proportional to the value of the instanton action and has a direct significance since $\operatorname{Im} t = R^2$, where R, in this context, is the radius of the Calabi–Yau manifold. The puzzle is that t and γ do not appear to be related in a simple way, though of course both are functions of ψ. It is perhaps worth recalling here the relations

$$t = -\frac{5}{2\pi i}\left\{\log\psi + \frac{\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{5})\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})\Gamma(n+\frac{4}{5})}{(n!)^4\psi^{5n}}\left[4\Psi(n+1) - \sum_{r=1}^{4}\Psi(n+r/5)\right]}{\displaystyle 5\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{5})\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})\Gamma(n+\frac{4}{5})}{(n!)^4\psi^{5n}}}\right\},$$

$$\frac{\gamma}{2\tan(2\pi/5)} - \frac{1}{2} = -\frac{5}{2\pi i}\left\{\log\psi + \frac{\displaystyle\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})}{(n!)^2\psi^{5n}}\left[2\Psi(n+1) - \sum_{r=2}^{3}\Psi(n+r/5)\right]}{\displaystyle 5\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{2}{5})\Gamma(n+\frac{3}{5})}{(n!)^2\psi^{5n}}}\right\},$$

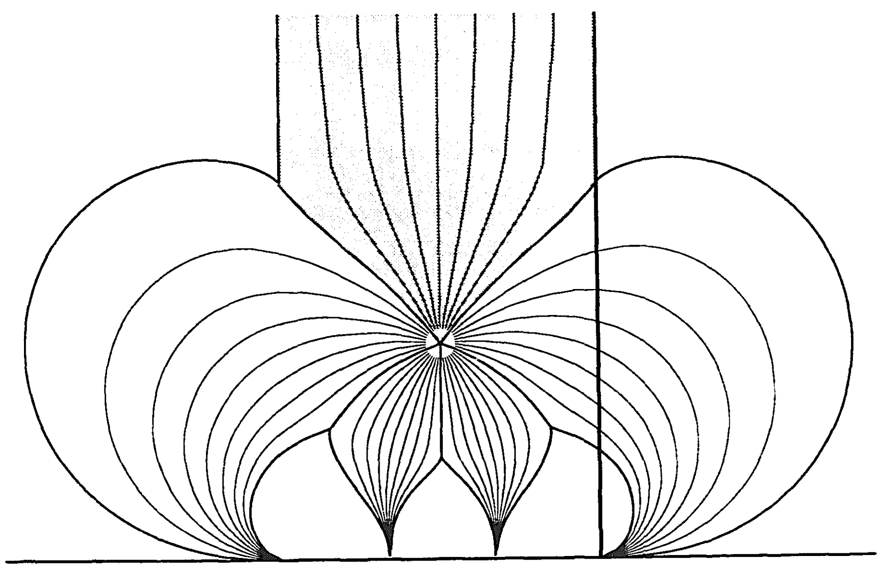

Fig. 12. The images of the lines $\arg\psi = \text{const.}$ in the t-plane. The shaded region is the image of the fundamental region and the other four regions are the images of the shaded region under the action of $\mathscr{A}$, $\mathscr{A}^2$, $\mathscr{A}^3$ and $\mathscr{A}^4$. Although it is not clear at this scale, the tangents to the boundaries coincide at the branch points, so the deficit angles are zero as are the angles at the tips of the images. The image of the point $\psi = 0$ is $t = -\frac{1}{2} + \frac{4}{5}i\sin^3(2\pi/5)$ and the four cusps touch the real axis at $t = 0, -\frac{1}{3}, -\frac{2}{3}, -1$.

to show that these functions definitely come from the same stable but are in fact different.

The transformation laws for t under the action of Γ follows from the first of equations (5.9). We observed previously that $t \to t-1$ under the action of $\mathcal{T}\mathcal{A}$. The transformation of t under the action of $\mathcal{A}$, say, is considerably more complicated. In fig. 12 we have plotted the images of the lines $\arg\psi = \text{const}$. The shaded region is the image of the fundamental region and the other four regions are the images of the fundamental region under $\mathcal{A}$, $\mathcal{A}^2$, $\mathcal{A}^3$ and $\mathcal{A}^4$. Since the tips of the latter four regions come down to touch the real axis we see that large R, that is large $\operatorname{Im} t$, is mapped to small R under the action of $\mathcal{A}$. The precise relation however is not as simple as $R \to 1/R$.

6. A mechanism for supersymmetry breaking

The purpose of this section is to present a speculative mechanism which breaks supersymmetry for the low-energy theory. The proposal is logically independent of the mirror symmetry that is the focus of the present article; however we include it here because it arises naturally as part of the discussion of the conifold at $\psi = 1$.

In refs. [17, 18] it has been observed that the nodes of a conifold can be resolved in different ways. One is by smoothing whereby the node is replaced by an S^3 as in fig. 1. Another way is to replace the node by an S^2 as indicated by fig. 13.

The construction that achieves this is entirely local in nature. However, in replacing the node by an S^2 we interfere with the cohomology group H^2 and it

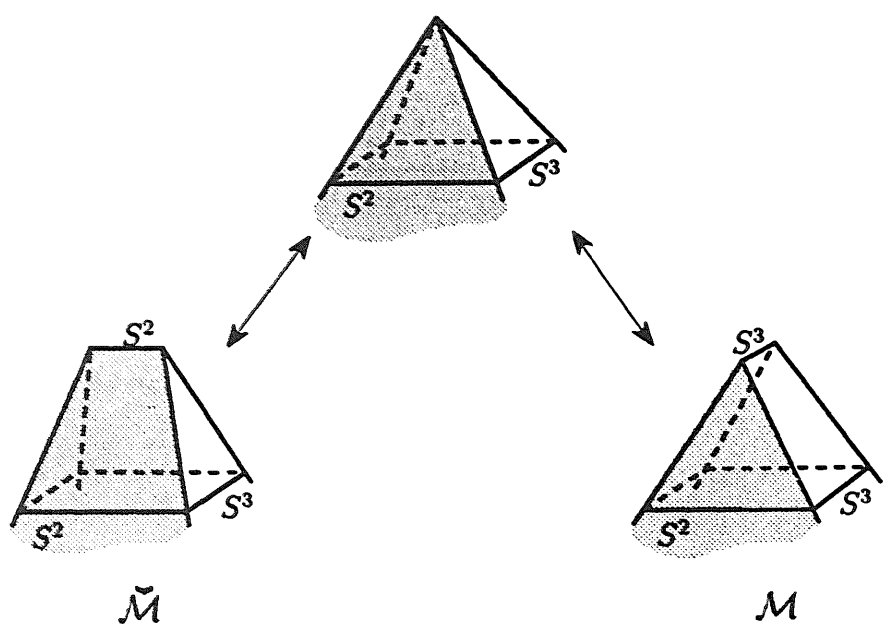

Fig. 13. Local neighborhoods of a node in a conifold, its small resolution in $\check{\mathcal{M}}$, and its deformation in $\mathcal{M}$. The conifold is singular while both $\check{\mathcal{M}}$ and $\mathcal{M}$ are smooth.

need no longer be the case that there remain any positive (1, 1)-forms. If this is so then the resulting manifold $\check{\mathcal{M}}$ is a Moishezon manifold but it is not Kähler. The case at hand is of just this type and this follows from the fact that the conifold has just one node. Let $\mathscr{C}$ be the resolving $S^2 = \mathbb{P}_1$ then the essential point is that $\mathscr{C}$ is a boundary (we are indebted to T. Hübsch for explaining this point to us). We can see this by considering the sequence of operations depicted in fig. 13. In passing from $\mathcal{M}$ to $\check{\mathcal{M}}$ a puncture is made in B_2 and the puncture is blown up into $\mathscr{C}$. Thus B_2 now has a boundary $\partial B_2 = \mathscr{C}$. Let also $J = ig_{\mu\bar{\nu}}\, dx^{\mu} \wedge dx^{\bar{\nu}}$ be the putative Kähler form, then we have

$$\int_{\mathscr{C}} J = \int_{\partial B_2} J = \int_{B_2} dJ .$$

The first integral is the area of $\mathscr{C}$ which is strictly positive. Thus dJ cannot vanish identically and hence $\check{\mathcal{M}}$ cannot be Kähler.

It was observed in ref. [32] that such a compactification of string theory has certain attractive features. The holonomy group of $\check{\mathcal{M}}$ is presumably O(6) so embedding the spin connection in the gauge group breaks $E_8 \times E_8$ to $E_8 \times O(10)$ and we can have broken supersymmetry while retaining vanishing cosmological constant. This proposal is cast in the geometrical language of Kaluza–Klein theory, because it is geometrical in nature and we do not yet know how to formulate these ideas in terms of conformal field theory. It does however show a way of resolving what was regarded as a difficult problem in the days of Kaluza–Klein theory which was how to introduce a length scale, different from the compactification scale, which could correspond to low-energy supersymmetry breaking. Here we have a second length scale arising as the radius of the resolving $\mathbb{P}_1$. In some appropriate sense supersymmetry is only slightly broken if this parameter is small, though we have no argument as to why the scale of the resolving $\mathbb{P}_1$ should be vastly different from the compactification scale.

It is a pleasure to acknowledge instructive discussions with Paul Aspinwall, Willy Fischler, Brian Greene, Vadim Kaplunovsky, Sheldon Katz, Andy Lütken, Fernando Quevedo, Graham Ross and Andy Strominger. We are indebted also to Vadim Kaplunovsky and Karen Uhlenbeck for making computer resources available.

Appendix A

A SECOND LOOK AT THE HOMOLOGY OF $\mathscr{W}$

The function of this appendix is to make a more detailed study of the homology of $\mathscr{W}$ and to close the gap left in sect. 3 by showing that the cycles A^2 and B_2 do indeed intersect in a point. We also make some further observations concerning the monodromy of the cycles and find cycles Q_j corresponding to the periods ϖ_j.

We begin by defining three-chains V_j, $j = 0, \dots, 4$,

$$\mathrm{V}_j(\psi) = \{x_k | x_5 = 1, \quad x_1, x_2, x_3 \text{ real and positive, the branch of } x_4 \text{ chosen such that } \arg x_4 \to \pi + 2\pi j/5 \text{ as } \psi \to 0\}.$$

The five V_j are chains rather than cycles, however a little thought shows that in virtue of the identities (2.2) they have a common boundary, that is $\partial \mathrm{V}_j$ is independent of j. So the difference of any two V_j is a cycle. We find x_4 by solving the quintic which we write in the form

$$x_4^5 - 5\psi x_1 x_2 x_3 x_4 + \Delta = 0, \qquad \Delta \stackrel{\text{def}}{=} 1 + x_1^5 + x_2^5 + x_3^5.$$

Setting also

$$x_4 = \Delta^{1/5}\eta, \qquad u = \frac{x_1 x_2 x_3}{\Delta^{4/5}},$$

we have

$$\eta^5 - 5\psi u \eta + 1 = 0. \tag{A.1}$$

This equation has of course five roots for given (x_1, x_2, x_3). For ψ sufficiently small these can be found by rewriting (A.1) in the form

$$\eta_j = -\alpha^j (1 - 5\psi u \eta_j)^{1/5}, \tag{A.2}$$

and iterating the equation. It is easy to show that u is bounded, in fact, u varies in the range

$$0 \leqslant u \leqslant 4^{-4/5}, \tag{A.3}$$

so for ψ sufficiently small the root η_j is unambiguously defined by the iteration. It is clear that for small ψ the V_j have no points in common apart from their boundaries.

We wish next to enquire how the V_j vary with ψ. In order to accomplish this we first consider the behaviour of the roots of eq. (A.1). It is evident that (A.1) cannot have a purely imaginary solution for η if ψ is real since then the first two terms would be purely imaginary and the third term real. Next we observe that (A.1) has a double root if in addition η satisfies the equation

$$\eta^4 = \psi u .$$

It follows that (A.1) has a double root if and only if

$$(\psi u)^5 = 4^{-4}, \tag{A.4}$$

and that a double root satisfies

$$\eta^5 = \tfrac{1}{4}. \tag{A.5}$$

One immediately sees that (A.1) cannot have a triple root or two double roots for any value of ψ. As a consequence of (A.3) and (A.4) η will have a double root for some value of (x_1, x_2, x_3) if and only if ψ^5 is real and $|\psi| \geqslant 1$. It follows that we can unambiguously extend the definition of the V_j to the ψ-plane cut as in fig. 3. Consider now the behaviour of the roots as ψ runs from 0 to ∞ through real values. For $\psi = 0$ the five roots are, as we have already seen, $\eta_j = -\alpha^j$. Since there are no purely imaginary roots for any real ψ, it follows that three of the roots have negative real part and two have positive real part. Moreover, since there cannot be a double root that is real and negative, in virtue of (A.5), the three roots with negative real part will always consist of a real root and a conjugate pair of complex roots.

On the other hand the two roots with positive real part will consist of a conjugate pair for

$$\psi < \frac{4^{-4/5}}{u},$$

and will be distinct and real for

$$\psi > \frac{4^{-4/5}}{u},$$

with a double root precisely when

$$\psi = \frac{4^{-4/5}}{u}.$$

The situation for large ψ was discussed in sect. 3. One of the roots approaches 0, while the other four recede to infinity along trajectories that are asymptotic to the four semi-axes. The two roots with positive real part therefore approach 0 and ∞, respectively, while the three roots with negative real part all become infinite. A

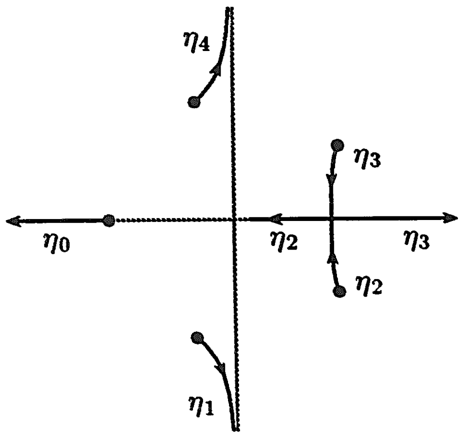

Fig. A.1. The trajectories of the roots of eq. (A.1) as ψ increases through real values. The roots η_2 and η_3 become real and move as shown if ψ has an infinitesimal positive imaginary part.

plot of the trajectories is presented in fig. A.1. These considerations will shortly allow us to compute $A^2 \cap B_2$.

We now look at the chains $V_j(\psi)$ for $\psi > 1$ which we define as the limit of $V_j(\psi)$ with Im ψ positive. It follows from the foregoing that V_0, V_1, V_4 and $V_2 \cup V_3$ are disjoint and that V_2 intersects V_3. Let X be the subset of $V_2 \cup V_3$ for which

$$\frac{4^{-4/5}}{\psi} \leqslant u\,. \tag{A.6}$$

X consists of two three-chains, corresponding to the two positive solutions of (A.1) that have positive real part, with their boundaries identified because there is a double root precisely when equality holds in (A.6). A little reflection should convince the reader that the two three-chains are topologically three-balls and that the boundaries of these three-balls are identified with opposite orientation so that X is an S^3. In fact X is the cycle A^2 of sect. 3.

We are now in a position to compute $A^2 \cap B_2$. The definition (3.1) of A_2 restricts all coordinates to be real and positive. The definition (3.4) of B_2 requires that $|x_1| = |x_2| = |x_3| = \delta$, so in fact

$$x_1 = x_2 = x_3 = \delta\,.$$

Because δ in the definition of B^2 is chosen to exclude multiple values of x_4 and in fact forces strict inequality in (A.6), there are exactly two points which satisfy all these restrictions, corresponding to the two positive real roots for η. However, as we have already seen, exactly one of these is on the branch for which $x_4 \to 0$ as $\psi \to \infty$. Consequently, the intersection of A^2 and B_2 consists of a single point.

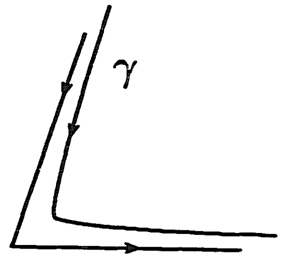

Fig. A.2. Two choices for the cycle γ_i. One composed of straight lines, the other smooth.

The last point we shall address in this part of the discussion is the monodromy of the cycles

$$V_{ij} \stackrel{\text{def}}{=} V_i - V_j ,$$

with respect to a loop circling the point $\psi = 1$. From the discussion above, it follows that as we go around such a loop, the chains V_j remain unchanged except for $j = 2, 3$, and that the chains V_2 and V_3 exchange the balls corresponding to eq. (A.6). This can be expressed as adding X to V_2 and substracting X from V_3 (there is an orientation of X implicit in this choice). The monodromy of the cycles can now be readily computed. We leave this as an exercise for the diligent reader.

CYCLES CORRESPONDING TO THE PERIODS ϖ_j

As a final topic in this appendix we wish to find cycles Q_j corresponding to the periods ϖ_j. The Q_j together with the relation (3.23) serve to define the cycles A^1 and B_1, which have hitherto been defined only implicitly. Of course we know that the period corresponding to $B_2(\psi)$ is proportional to $\varpi_0(\psi)$, so cycles corresponding to the ϖ_j are proportional to $B_2(\alpha^j\psi)$, with the continuation performed along paths that go from ψ to $\alpha^j\psi$ without crossing the cuts. We shall here give an alternative definition of the Q_j and relate these cycles to the V_j discussed above.

Let γ_i, $i = 1, 2, 3$, be the one-cycles consisting of the union of the two half-lines $\arg x_i = 2\pi/5$ and $\arg x_i = 0$. The reason for choosing the cycle as we have is that u is bounded on the cycle. For other cycles, such as taking the x_i to be real, this is no longer true so that the term ψu in eq. (A.2) is not necessarily small for small ψ. We define the cycle Q_k by taking $(x_1, x_2, x_3) \in \gamma_1 \times \gamma_2 \times \gamma_3$ and x_4 to be given by the $(k+3)$rd branch of the quintic.

Solving eq. (A.2) by iteration we find

$$\eta_k^{(0)} = -\alpha^k ,$$

$$\eta_k^{(1)} = -\alpha^k\left(1 + \alpha^k\psi u\right),$$

$$\eta_k^{(2)} = -\alpha^k\left(1 + \alpha^k\psi u - \left(\alpha^k\psi u\right)^2\right), \qquad \text{etc.}$$

The important point is that

$$\eta_k(\psi)=\alpha^k\eta_0(\alpha^k\psi). \tag{A.7}$$

Let $q_k(\psi)$ be the period evaluated on Q_k,

$$q_k(\psi)=\psi\int_{\gamma_1\times\gamma_2\times\gamma_3}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3}{\Delta^{4/5}\left(\eta_{k+3}^4-\psi x_1x_2x_3\Delta^{-4/5}\right)}. \tag{A.8}$$

In virtue of eq. (A.7) we see that

$$q_k(\psi)=q_0(\alpha^k\psi),$$

so it is sufficient to study q_0. The factor $(\eta^4-\psi x_1x_2x_3\Delta^{-4/5})^{-1}$ can be expanded as a power series in ψ,

$$\left(\eta_3^4-\psi x_1x_2x_3\Delta^{-4/5}\right)^{-1}=\alpha^3\sum_{n=0}^{\infty}c_{n+1}\left(\frac{5\alpha^3\psi x_1x_2x_3}{\Delta^{4/5}}\right)^n.$$

Substituting this expression into eq. (A.8) we have

$$q_0(\psi)=\tfrac{1}{5}\sum_{m=1}^{\infty}c_mI_m\alpha^{3m}(5\psi)^m, \tag{A.9}$$

with

$$I_m=\int_{\gamma_1\times\gamma_2\times\gamma_3}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3(x_1x_2x_3)^{m-1}}{\Delta^{4m/5}}.$$

The integrals I_m can be evaluated in closed form. First observe that

$$\int_{\gamma_1}\frac{\mathrm{d}x_1\,x_1^{m-1}}{\Delta^{4/5}}=(1-\alpha^m)\int_0^{\infty}\frac{\mathrm{d}x_1\,x_1^{m-1}}{\Delta^{4/5}},$$

so we have

$$I_m=(1-\alpha^m)^3\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3(x_1x_2x_3)^{m-1}}{\Delta^{4m/5}}. \tag{A.10}$$

On introducing new variables $y_i=x_i^{5/2}$ and then going over to polar coordinates, we find that the integral factorizes into three integrals that are easily evaluated in terms of B-functions. The result is

$$I_m=(-1)^{m+1}\left(\frac{2\pi i}{5}\right)^3\alpha^{-m}\frac{\Gamma(m/5)}{\Gamma(4m/5)\Gamma^3(1-m/5)}, \tag{A.11}$$

where in writing this last relation we have used the fact that $\alpha^{1/2} = -\alpha^3$. Substituting eq. (A.11) into eq. (A.9) we find

$$q_0(\psi) = \left(\frac{2\pi i}{5}\right)^3 \tfrac{1}{5} \sum_{m=1}^{\infty} (-1)^{m+1} c_m \alpha^{2m} \frac{\Gamma(m/5)}{\Gamma(4m/5)\Gamma^3(1-m/5)} (5\psi)^m .$$

When the coefficients c_m are calculated we find

$$c_m = (-1)^{m+1} \frac{\Gamma(4m/5)}{\Gamma(m)\Gamma(1-m/5)} ,$$

yielding

$$q_0(\psi) = \left(\frac{2\pi i}{5}\right)^3 \frac{1}{5} \sum_{m=1}^{\infty} \frac{\alpha^{2m}\Gamma(m/5)(5\psi)^m}{\Gamma(m)\Gamma^4(1-m/5)} .$$

We recognize that, in virtue of eq. (3.15), the right-hand side of this relation is proportional to ϖ_0. In fact

$$q_0(\psi) = -\left(\frac{2\pi i}{5}\right)^3 \varpi_0(\psi) .$$

Finally, we observe from eq. (A.10) that the Q-cycles are related to the V-chains by

$$\mathrm{Q}_j = (1-\mathscr{A})^3 \mathrm{V}_{j+3} ,$$

with $\mathscr{A}$, as previously, the operation that replaces ψ by $\alpha\psi$. Thus we have

$$\mathrm{Q}_j = \mathrm{V}_{j-2} - 3\mathrm{V}_{j-1} + 3\mathrm{V}_j - \mathrm{V}_{j+1} .$$

Appendix B

FURTHER PROPERTIES OF THE PERIODS

We record here some further results pertaining to the periods $\varpi_j(\psi)$. Recall that $\varpi_j(\psi) = \varpi_0(\alpha^j\psi)$, so for $|\psi| < 1$ we have, from eq. (3.15),

$$\varpi_j(\psi) = -\tfrac{1}{5} \sum_{m=1}^{\infty} \frac{\alpha^{2m}\Gamma(m/5)(5\alpha^j\psi)^m}{\Gamma(m)\Gamma^4(1-m/5)} , \qquad |\psi| < 1 . \tag{B.1}$$

Our main purpose here is to obtain explicit expressions for the $\varpi_j(\psi)$ valid throughout the fundamental region. It is simplest to begin by discussing the basis (3.13). It is perhaps notationally simpler to regard the hypergeometric function that appears on the right-hand side of eq. (3.13) as a ${}_5F_4$ which has $a_5 = c_4 = 1$. We choose the basis

$$\tilde{\varpi}_k(\psi) = \frac{\Gamma^5(k/5)}{\Gamma(k)}(5\psi)^k {}_5F_4\left(\frac{k}{5}, \frac{k}{5}, \frac{k}{5}, \frac{k}{5}, 1; \frac{k+1}{5}, \frac{k+2}{5}, \frac{k+3}{5}, \frac{k+4}{5}; \psi^5\right)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma^5(n+k/5)(5\psi)^{5n+k}}{\Gamma(5n+k)}, \qquad |\psi| < 1,$$

the last equality following in virtue of the multiplication formula (3.12). To analytically continue these functions we write them as integrals,

$$\tilde{\varpi}_k(\psi) = -\int_C \frac{\mathrm{d}s}{(\mathrm{e}^{2\pi i s} - 1)} \frac{\Gamma^5(s+k/5)(5\psi)^{5s+k}}{\Gamma(5s+k)}, \qquad 0 \leqslant \arg\psi \leqslant 2\pi/5.$$

For $|\psi| > 1$ the contour can be closed to the left. Note that there are no poles when $s = -N$, $N = 0, 1, \ldots$, for then $5s + k = k - 5N$ and the factor $1/\Gamma(5s+k)$ renders the integrand finite. There are however fourth-order poles when $s = -N - k/5$. These are not of fifth order, again because of the $\Gamma(5s+k)$ in the denominator. Extracting the residues involves the expansion of ψ^{5s} about $s = -N - k/5$. This produces the $\log\psi$, $\log^2\psi$ and $\log^3\psi$ terms. The ϖ_j are given in terms of the $\tilde{\varpi}_k$ by the relation

$$\varpi_j(\psi) = -\frac{1}{80\pi^4} \sum_{k=1}^{4} \alpha^{jk}(\alpha^k - 1)^4 \tilde{\varpi}_k(\psi).$$

The upshot is that the ϖ_j have series expansions of the form

$$\varpi_j(\psi) = \sum_{r=0}^{3} \log^r(5\psi) \sum_{n=0}^{\infty} b_{jrn} \frac{(5n)!}{(n!)^5(5\psi)^{5n}}, \qquad |\psi| > 1,$$

with the coefficients b_{jrn} given by somewhat lengthy expressions. Defining

$$S_{jm} = \sum_{k=1}^{5} \alpha^{k(j+1)}(\alpha^k - 1)^m, \qquad \Phi(z) = \Psi(1+z) - \Psi(1+5z),$$

we have

$$b_{j0m} = -\frac{1}{6(2\pi i)^3}\Big\{6(2\pi i)^3(S_{j0}-1) + 6(2\pi i)^2[2\pi i + 5\Phi(m)]S_{j1}$$

$$+3(2\pi i)\big[2(2\pi i)^2 + 5(2\pi i)\Phi(m) - 5\Phi'(m) + 25\Phi^2(m)\big]S_{j2}$$

$$+5\big[\Phi''(m) + 5(2\pi i)^2\Phi(m) - 15\Phi(m)\Phi'(m) + 25\Phi^3(m)\big]S_{j3}\Big\},$$

$$b_{j1m} = -\frac{5}{2(2\pi i)^3}\Big\{2(2\pi i)^2 S_{j1} + (2\pi i)[2\pi i + 10\Phi(m)]S_{j2}$$

$$+\tfrac{5}{3}\big[(2\pi i)^2 + 15\Phi^2(m) - 3\Phi'(m)\big]S_{j3}\Big\},$$

$$b_{j2m} = -\frac{25}{2(2\pi i)^3}\{2\pi i S_{j2} + 5\Phi(m)S_{j3}\},$$

$$b_{j3m} = -\frac{125}{6(2\pi i)^3}S_{j3}.$$

Finally we record the series expansion of z^2 about $\psi = 1$,

$$z^2(\psi) = \frac{4\pi^2}{5^{3/2}}\sum_{m=0}^{\infty} a_m(1-\psi^5)^{m+1}, \qquad |\psi^5 - 1| < 1,$$

where the coefficients satisfy the recurrence relation

$$0 = 625m^2(m^2-1)a_m - 125m(m-1)(20m^2-40m+23)a_{m-1}$$

$$+125(m-1)(30m^3-150m^2+261m-157)a_{m-2}$$

$$-(2500m^4-22500m^3+76625m^2-116875m+67226)a_{m-3}$$

$$+625(m-3)^4 a_{m-4}$$

with the initial values

$$a_0 = -\tfrac{1}{5}, \qquad a_1 = -\tfrac{3}{50}.$$

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